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ABSTRACT

The purpose of this study was to uncover information about and gain a greater insight into the extent to which students who are successful in a basic algebra course: 1) demonstrate a reversibility of reasoning processes when solving algebraic problems; 2) demonstrate a flexibility of reasoning processes when solving algebraic problems; 3) generalize reasoning processes and algebraic operations; and 4) transfer reasoning processes and algebraic operations within the content of algebra. A Soviet "ascertaining" methodology was chosen, with this clinical approach viewed as appropriate to investigate the algebra student's thought processes. Six instruments were used to provide descriptions of the subjects. Four females were selected non-randomly for the study. Data indicated that the subjects chosen varied widely on measures of the four processes of understanding identified. Tests based on measures of instrumental understanding did not accurately reflect a student's relational understanding. Further, these students possessed definite styles of reversibility and many of the same error patterns as less successful students. It is felt that this study provides an exploration of the elusive term, "understanding" of mathematics, which mathematics educators have long noted as an important thing for students to develop. (MP)

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# PROCESSES USED BY COLLEGE STUDENTS IN UNDERSTANDING BASIC ALGEBRA

by

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B.S., Shippensburg State College

M.A., Lehigh University

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of the

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Sidney Lee Rachlin

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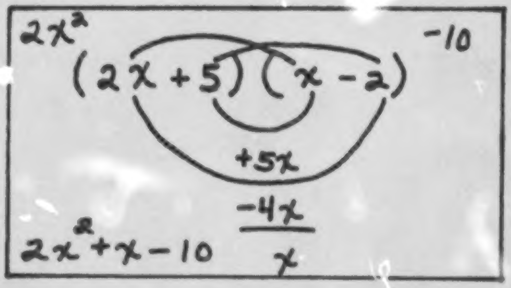
# CHAPTER I

# THE PROBLEM AND ITS BACKGROUND

## Introduction

What does it mean to say that a student understands algebra? Suppose a student has been given the task of multiplying two binomials, say 2x + 5 and x - 2. After a moment the student hands in his paper (suppose he has written the expressions enclosed in the box below). Upon request, the student explains how he obtained his product: "First I wrote the 2x + 5 and the x - 2 down side-by-side. Then I just used FOIL." The student, upon being pressed for a further explanation of FOIL, explains: "First you multiply the first terms -- that's 2x times x is $2x^2$. Then you multiply the outside terms -- that's 2x times -2 is -4x. Next you multiply the inside terms -- that's +5 times x is +5x, and combine terms. That's x. Finally you multiply the last terms -- that's +5 times -2 is -10. Then you put them all together and you get $2x^2 + x - 10$." When asked if he understands multiplication of binomials, the student stares at the paper and questions,



"Did I make a mistake?" The student has provided an indication that he understands how to multiply binomials. He has also shown that he knows he used the FOIL method.

Suppose a second student has been given the same task and her work is that reproduced in the box below. Her explanation was: "First you write down the problem. Then you multiply x - 2 times the 2x and x - 2 times the +5. Next you multiply the 2x times the x - 2 and you get $2x^2 - 4x$. Then you multiply +5 times the x - 2 and you get 5x - 10. Combine similar terms and you have $2x^2 + x - 10$." When asked why she did it that way, she replied, "That's the distributive property." This student has also provided an indication that she understands how to multiply binomials. She has shown that she knows she used the distributive property. Could the responses of these two students be rote? Are these responses sufficient evidence to say that either student understands multiplication of binomials?

$$\boxed{\begin{array}{c} (2x+5)(x-2) = \\ 2x(x-2) + 5(x-2) = \\ 2x^2 - 4x + 5x - 10 = \\ 2x^2 + x - 10 \end{array}}$$

The great majority of the exercises in any algebra textbook may be solved with only a "how to" understanding of algebraic procedures and generalizations. The student recognizes that the task fits an algorithm and applies the algorithm. For many students, and perhaps many teachers, success on a set of such exercises is taken as evidence that one understands the algebraic procedure or generalization.

Another view of what it means for a student to understand has been proposed by Holt (1964):

> It may help to have in our minds a picture of what we mean by understanding. I feel I understand something if and when I can do some, at least, of the following: (1) state it in my own words; (2) give examples of it; (3) recognize it in various guises or circumstances; (4) see connections between it and other facts or ideas; (5) make use of it in various ways; (6) foresee some of its consequences; (7) state its opposite or converse. The list is only a beginning; but it may help us in the future to find out what our students really know as opposed to what they can give the appearance of knowing, their real learning as opposed to their apparent learning. (pp. 136-137)

Holt suggests that the question of whether or not a student understands is not a dichotomy. Rather, each student's understanding should be viewed as having several different aspects.

In the case of the hypothesized students' understanding of the multiplication of binomials, one might try to ascertain if either sees a relationship between the two approaches. Could either use the other's method of multiplication? Could either multiply the binomials if they were written vertically? Could either apply the left-hand distributive property as well? How would they handle a task in which an algorithm other than the one they normally use was more efficient? In Holt's terms, could they "recognize [the appropriateness of another algorithm] in various guises and circumstances" and "make use of it in various ways"? Expressed another way, to what extent are they flexible in their approach?

The first student showed that he could recognize the appropriateness of and apply the "FOIL" algorithm, but could he generate his own appropriate tasks? For example, if asked to make up a problem "like" the given one, could he generate a similar problem? If he did, could he generate another, a little more complex? How would his examples vary? To what degree of variation would he extend his examples: $(2a + 5)(a - 2)$? $(4xy - z)(3x - z)$? $(4x^m y^{m+1} - 3)(3x^m y^{m+1} - 4)$? Would he at some point make a statement explaining in general what the product was like? Would he, in Holt's terms, "give examples of [his generalization]" and "state it in his own words"? Or expressed another way, to what degree can he generalize algebraic processes and operations?

Suppose the second student demonstrates that she can also factor the trinomial, $2x^2 + x - 10$. Does she see a relationship between multiplying the original binomials and factoring the indicated trinomial? What process would she use to find the binomial that multiplied by $2x + 5$ yields the product $2x^2 + x - 10$? Would she work backward from the $2x^2 + x - 10$ (search for the binomial to multiply by $2x + 5$) or divide $2x^2 + x - 10$ by $2x + 5$? Could she solve the problem either way? Would she, in Holt's terms, be able to "state its opposite or converse"? Could she apply her algorithm in reverse order? Expressed in other terms, to what degree does she demonstrate a reversibility of reasoning processes when applying algebraic processes and operations?

Would the students be able to satisfy Holt's two remaining conditions? Would they "see connections between it and other facts or ideas" and "foresee some of its consequences"? To what extent would they recognize a relationship between multiplying $(2x + 5)(x - 2)$ and multiplying $(2\sqrt{3} + 5)(\sqrt{3} - 2)$ or $(2 + 5i)(3 - 2i)$? How would they find two binomials whose product was a binomial? Whose product had four terms? How quickly would they realize that finding two binomials whose product had five terms is impossible? In other words, to what extent do they transfer algebraic processes and operations to problems not specifically taught in class?

Davis (1978) has called these extensions of the meaning of understanding procedures and generalizations, "why to" understanding. Skemp (1976) described the differences in assessing instrumental ("how to") and relational ("why to") understanding. As an indicator of a student's knowing "how to" (instrumentally) apply algorithms teachers generally rely on the student's answers to algebraic tasks. At times these measures are reinforced by requesting that specific steps of the algorithm be recorded, but these steps may become automatic and meaningless. In investigating a student's understanding of procedures and generalizations, it is more appropriate to focus on the mental processes that direct the student's search for the solution. As Wittrock

(1973) says, "It may seem trivial to focus on the details, but it is through the study of these specific processes that a deeper understanding of the learning of mathematics will develop" (p. 29).

## Statement of the Problem

The purpose of this investigation was to uncover information about and gain a greater insight into the extent to which students who are successful in a basic algebra course:

(1) Demonstrate a reversibility of reasoning processes when solving algebraic problems.

(2) Demonstrate a flexibility of reasoning processes when solving algebraic problems.

(3) Generalize reasoning processes and algebraic operations.

(4) Transfer reasoning processes and algebraic operations within the content of algebra.

The investigation of four processes of understanding was designed to go beyond product measures of students' learning to their internalized thought processes. With all its inherent limitations, a clinical approach has been recognized by mathematics educators as the most appropriate methodology for investigating internalized operations of thought (see, for example, Hatfield, 1979; Kantowski, 1974; Kilpatrick, 1969; Lester, 1980). A clinical methodology was used in the investigation in which the protocols of four subjects solving algebra problems were analyzed in detail for regularities in observable sequences of behavior. Analyses were based on the qualitative evaluation of audio-recorded interviews in which the subjects "thought aloud" as they solved problems related to the algebra content they were learning concurrently in a basic algebra course. Hints were readily provided in an effort to "catch" the development of the thinking process. The subjects were university students enrolled in a non-credit basic algebra course because they had failed an examination to exempt the course. The subjects were identified by their basic algebra instructors as above average.

The task remaining in this first chapter is threefold. First, a theory of the understanding of algebra is developed and related to the classic studies of meaning in elementary school mathematics and to more recent literature on algebraic abilities. Next, the terms "instrumental understanding" and "relational understanding" are defined. Finally, questions drawn from the theoretical background are posed to give direction to the qualitative analysis.

## Theoretical Background

Throughout much of this century the teaching of mathematics has been dominated by two theories of learning: connectionist learning theory and meaningful learning theory. The transition among educators from support for the connectionist theory of Thorndike to support for the meaningful learning theory of Brownell has been detailed by Weaver and Suydam (1972). Brownell (1954) categorized this transition as an evolution from a predominance of psychological research centered on the products of learning toward a more direct study of the process of learning.

Weaver and Suydam (1972) point out that meaningful learning theory was rarely explicated at the secondary school level and "rarely was there related research conducted within the context of secondary-school mathematics" (p. 2). Hence the contrasts between meaningful and connectionist learning theories were made almost exclusively at the elementary school level. It was generally assumed that theory and research into the learning of arithmetic would apply readily to the learning of algebra (see, for example, Thorndike, 1926; Judd, 1939). The following historical perspective is presented as a foundation for the theoretical constructs of understanding.

## A Historical Perspective

### Connectionist Learning Theory

The connectionist learning theory has also been known as the associationist, rote, or drill theory of learning. In The Psychology of Algebra, Thorndike (1926) presented a connectionist theory of the learning of algebra:

> Learning algebraic computation is, and should be, in large measure the formation and organization of a hierarchy of mental connections or bonds. The science of teaching algebra must consider what these bonds are, the amount of

> practice each should have, how this practice should be distributed, how the order and methods of formation of these bonds may provide the maximum of facilitation and the minimum interference among them. (p. 250)

To the connectionist theorist learning algebra consists of learning a series of rules to apply. The successful student is one who makes the proper connections at the proper times. Perception and memory are closely tied to this view of success. Applying a correct series of connections at the proper time implies the ability to perceive the structure of a problem, choose the appropriate series of connections, and produce all of the connections in the series.

Thorndike further characterized the general psychology and pedagogy of the connectionist approach to thinking as "the same for algebra as for arithmetic" (p. 250). The nature of the connectionist theory of learning algebra, and its comparison with the meaningful learning theory, must be inferred from the literature on the learning of arithmetic.

Brownell (1935) identified four major criticisms of the connectionist theory of learning. These were the prohibitive task of memorizing multitudes of bonds, the invalidity of product measures as accurate measures of process, the invalidity of the analysis of adults' performance of arithmetic, and the negative effect the connectionist theory may have on the later learning of mathematics.

Brownell's four major criticisms of the connectionist theory were made in reference to the learning of arithmetic. Parallel criticism might be voiced in terms of a strictly rote learning of algebra.

1. Magnitude of the task of rote learning. Algebra could be taught as a series of meaningless rules. If it were taught in such a manner, the demands on the student's memory would be excessive. Each operation in each new context would necessitate the formation of new bonds. As the number of rules grows and task environments become more complex, the tendency to confuse series of steps increases. For example, the rules for simplifying algebraic fractions with monomial numerators and denominators become entwined with the rules for algebraic fractions when either the numerator or denominator is not a monomial.

2. The invalidity of product measures as accurate measures of process. Chazal (cited in Brownell, 1935) found that although a teacher had attempted to drill the basic facts of arithmetic without understanding, the students developed their own ways of thinking of the combinations. Had Chazal not used an interview technique with the children, it might have been incorrectly assumed that nearly half of the children's responses were based on rote-learned connections.

If success is measured by the students' ability to recognize the appropriateness of algorithms and apply the algorithms, one might expect that the error patterns of the less successful students would not be found for the successful students. Are successful students "really" free of these error patterns? Would interviewing yield a more accurate reflection of the nature of their understanding?

3. The invalidity of the analysis of adult's performance of algebra. The learning of intermediate algebra has been ordered in textbooks in a logical progression of readily transferable topics. For example, the basic operations with radical expressions and complex numbers follow the basic operations with polynomials. Do the students see the connection? Do they transfer the processes and operations learned with polynomials to the learning of operations and processes for radical expressions and complex numbers? Even the awareness of such basic notions as the relationship between multiplication of polynomials and factoring polynomials, while natural for the adult (expert), might not be natural for the learner. How do students view the connections between the processes and operations they have learned?

4. The negative effect a rote learning of algebra may have on the later learning of mathematics. Algebra has a reputation for being difficult to learn. Its abstract nature may naturally lead the unsuccessful student toward a negative view of mathematics. But what of the successful student? How does the successful student view the learning of mathematics?

## Meaningful Learning Theory

Meaningful learning theory, also known as the meaning theory of arithmetic, evolved from the criticism of purely rote learning. The most widely accepted formulation of meaning theory was framed by Brownell (1935), who described the meaning theory of arithmetic in the following paragraph:

> This theory makes meaning, the fact that children see sense in what they learn, the central issue in their arithmetic instruction. Drill is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. The "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. (p. 19)

Brownell (1945) identified four reasons why meanings must be taught in arithmetic: "(1) arithmetic can function in intelligent living only when it is understood; (2) meanings should facilitate learning; (3) meanings increase the chances of transfer; and (4) meaningful arithmetic is better retained and is more easily rehabilitated than is mechanically learned arithmetic" (p. 494).

Along with these reasons why meanings must be taught, Brownell (1948) proposed three criteria of learning to be used in educational research: changes in the level of process, transfer, and retention. For research relating to complex learning tasks, these criteria were intended to supplement and at times even supplant the criteria of rate and accuracy.

The processes of understanding -- analysis, synthesis, generalization, discrimination, and comparison -- were hypothesized by Brownell (1946) to vary in definiteness and completeness.

> All learning produces some understanding. Since understanding in some measure accompanies all learning, the problem of education cannot be defined in the creation of understanding. Some degree of understanding is inevitable if a child learns at all. The problem of education, rather, is to make sure that children possess the kinds and degrees of understanding which are essential to intelligent behavior consistent with their needs (p. 31).... Understanding of a principle means that one can identify its appropriateness and usefulness in situations where it has not been seen or used previously. Understanding of a process implies that one knows when and how to use it effectively. (p. 41)

It was the duty of educational researchers to include measures of change in understanding in their studies of arithmetic learning.

Brownell (1939) also applied the criterion of transfer in two ways in his study of the value of the "crutch" in subtracting with regrouping. The first evidence of transfer was seen in the renewed application of the crutch when children extended subtraction with regrouping to involve multiple borrowing (a generalization of subtraction). The second evidence of transfer was seen in the renewed application of the crutch to new applications of the algorithm such as in problem solving and division. As a justification for his inclusion of transfer as a criterion of learning, Brownell (1948) stated that the "ability to use the learned idea or skills in situations differing from the learning situation is evidence of understanding; and the greater the differences in situations and the longer the transfer measures are deferred, the better the evidence is" (p. 181). The degree of transfer gives the educational researcher a measure of the understanding that has developed (Brownell, 1946).

Brownell (1948) viewed the criterion of retention in the broadest sense of the word. Used precisely, retention refers to the maintenance of skill, knowledge, or attitude without practice of any kind after the period of learning. But in the learning of mathematics, as well as most school subjects, the importance of retention is that what is taught in the classroom is rarely taught for temporary purposes. Brownell's use of the term "retention" was broadened to include implied and embedded practice. What is learned in the classroom is generally intended for use in later situations. By including the criterion of retention in his study of the value of the "crutch" in subtracting with regrouping, Brownell (1939) was able to determine that the crutch did not entirely disappear even for children who seemed to have dropped it completely. Instead, he found that in the absence of practice children would revive the crutch until they were again comfortable with the algorithm.

In a study of the relative merits of the equal addends and the decomposition methods for subtraction with regrouping as mechanical skills and as rational activities, Brownell and Moser (1949) applied six measures of evaluation: rate, accuracy, smoothness of thought processes or work habits, understanding of procedures, transfer, and retention. Brownell and Moser demonstrated the significant superiority of the meaningful application of the decomposition method on the criteria of transfer and retention.

The three additional criteria of learning proposed by Brownell (changes in the level of processes, retention, and transfer) may also be applied to the learning of algebra. How do students' solution processes change while they are in the midst of learning algebra? Do they alter the level of their processes of understanding? How do students solve problems that recur in other tasks? For example, does a student subtract polynomials differently when they are the numerators of algebraic fractions? To what degree do students transfer algebraic skills within the content of algebra?

Brownell agreed with Holt; the question of understanding was not dichotomous. Understanding varied in the degree of definiteness and completeness. In order to classify the component abilities of relational understanding the discussion of the theoretical background next focuses on the model of mathematical abilities developed by Krutetskii (1976).

## Krutetskii's Model of Mathematical Abilities

Krutetskii (1976) proposed an information processing model of the mathematical abilities of school children based on his 12 years of research. The research leading up to the formation of the model, as well as later research on the application of the model, is described in Chapter II. The present account of Krutetskii's writings is limited to comments relevant to the four processes of understanding investigated in the present study. Three of these processes were included directly in Krutetskii's model: "the ability for rapid and broad generalization of mathematical objects, relations, and operations; ... the flexibility of mental processes in mathematical activity; ... [and] the ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning)" (p. 350). The fourth process, an ability to transfer mathematical operations and processes to new settings, although not directly included in the model, can be discussed in terms of it.

### Reversibility of Mental Process

In his studies of mathematical abilities Krutetskii (1969b, 1976) identified "the ability to establish 'reversible' (two-way) associations in the process of learning mathematical material" as an essential aptitude for the mastery of mathematics and, in particular, for the formation of algebraic concepts. Krutetskii (1969b) defined the basic concept of "reversibility" as follows:

> By reversible (two-way) associations (and series of associations) we mean those associations in which the thought or realization of the first element evokes the thought or realization of the second element (or of the last element in a series of associations); and the thought or realization of the second element (or of the last element) evokes the thought or realization of the first element (associations of the type A--E). (p. 51)

In a more general sense, Krutetskii (1976) explained "reversibility" as "an ability to restructure the direction of a mental process from a direct to a reverse train of thought" (p. 143). In the transition from a theorem to its converse, for example, Krutetskii indicated that the thought did not necessarily have to travel precisely the same route, but simply moved in a reverse direction.

Krutetskii's discussion of reversibility suggests the importance of this ability to a meaningful understanding of algebra. For students of algebra "to see the big picture," they must be able join together the algebraic operations they have learned. Krutetskii's research suggests that students may be able to multiply polynomials (forward thinking) and to factor polynomials (reverse thinking), without connecting the two.

This concept of reversibility is similar to an adaptation of Piaget's "operational reversibility" in the work of Johannot (1974). Johannot, a student of Piaget, applied Piaget's concept of reversibility in developing a theory of adolescent stages of reversibility based on tasks involving the meaning of complex fractions and the solution of linear equations. This research will be discussed in greater detail in a later section.

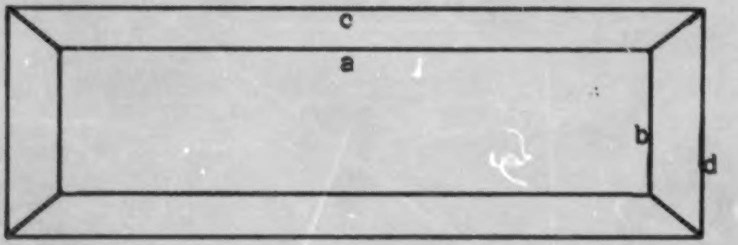
## Flexibility of Mental Process

Flexibility was identified by Krutetskii (1976) as the ability to switch from one level of thinking of a problem to another. Kilpatrick (1978) noted that there were two aspects of flexibility included in Krutetskii's research: flexibility that can be shown either within or across problems. The first aspect concerned a flexibility of approach in moving from problem to problem -- the degree to which a successful solution process on a previous problem did not "fix" a student's approach to a subsequent problem. The second aspect of flexibility concerned the ease with which a student switched from one method of solving a problem to another method of solving the same problem.

Krutetskii's description of flexibility is compatable with Duncker's concept of "functional fixity." Duncker (1945) attributed much of an individual's failure to solve problems to an inability to reorganize perceptions to see new functions for available material. Lack of flexibility might also be compared in many ways to the Gestalt idea of "set" (Einstellung), the state of preparedness for a particular task that is brought about by instruction, previous experience, or general circumstances. Wertheimer (1959) provided an example of the lack of flexibility ("functional fixity" or "set") which some students demonstrated in moving from a stereotype solution. After a lesson on finding the areas of trapezoids by using the formula:

$$A = \frac{a + b}{2} h$$

the teacher asked the students to calculate the area of the frame around a picture of the type given below.



While many students worked busily calculating the areas of the trapezoids composing the picture frame, a few subtracted ab from cd.

Krutetskii's two forms of flexibility are readily applicable to an investigation of algebraic processes. The ability to switch from one approach to another, more efficient, approach is a question of degree. One can assume that the successful student can find an approach to solve problems closely related to the curriculum, but will that approach be the most efficient one in terms of ease and accuracy? Will the student who finds a solution to a problem be able to find other approaches for obtaining the solution?

## The Ability to Generalize

Krutetskii's characterization of the ability to generalize mathematical material is very much in line with Bartlett's. Bartlett (cited by Dienes, 1965) distinguished two kinds of generalization in thinking. The first was that once a set of varying examples of a generalization has been accepted, then the further extensions of this set will be more likely to be accepted. This corresponds to Dienes's (1965) characterization of generalization as an extension of an already formed class. This notion of generalization is commonly applied in the ordered series of exercises found in most mathematics textbooks in which increasingly more complicated extensions of a generalization are made.

The second type of generalization in thinking identified by Bartlett (cited in Dienes, 1965) was the "process of extracting the common properties from different instances and classing these together. This process of generalization is assumed to include some 'stamping in' of the common properties and some 'stamping out' of the uncommon properties, that is those that are not to be generalized" (Chapter 2, p. 15). This second type of generalization is closely related to Dienes's (1965) definition of abstraction as a process of class formation.

Krutetskii (1976) considered the ability to generalize mathematical material to be on two levels:

> (1) as a person's ability to see something general and known to him in what is particular and concrete (subsuming a particular case under a known general concept), and (2) the ability to see something general and still unknown to him in what is isolated and particular (to deduce the general from particular cases, to form a concept). (p. 237)

Krutetskii's definition of the ability to generalize makes the ability to generalize mental processes and algebraic operations to a large measure synonymous with the ability to transfer mental processes and algebraic operations. The difference is in the level at which one sets a psychological lens. In addition to an ability to generalize algebraic skills and operations within each topic of algebra, a meaningful understanding of algebra should include the "generalization" of these generalizations. For example, the student who has learned the basic operations for polynomials should readily generalize these operations in learning the operations for radical expressions and complex numbers. In this study, the ability to generalize will refer to generalizations made within specific topics (e.g., polynomial expressions, radical expressions, and complex numbers) and the ability to transfer will refer to generalizations made between topics. In this regard, the ability to transfer may be thought of as a second level of the ability to generalize.

It might be assumed that successful students have, by definition, an ability to apply algebraic operations to appropriately recognized tasks. The main aspect of the ability to generalize of interest in this study was the students' ability to extend a class to unknown (student generated) variants. How do successful students generalize their solutions? What do they view as more complicated examples of the same type? How general is their response?

Kilpatrick (1978) emphasized one major distinction between Krutetskii's notion of generalization and that of most other educational psychologists. Krutetskii (1976) showed that the ability to deduce the general from particular "cases", may in fact be limited to one case.

> The method of gradual generalization is not the only way to a mastery of knowledge about mathematics; there is another way that differs from it in principle but that leads to the same result. Along with the method of gradual generalization of mathematical material on the basis of variations in a diversity of particular cases (the method for most pupils), there is another way, in which able pupils, without special exercises or hints from the teacher, independently generalize mathematical objects, relations, and operations "on the spot," on the basis of an analysis of just one phenomenon, into a number of similar phenomena. (pp. 261-262)

The concept of generalizing from a single example has also been suggested by Freudenthal (1978) and Wertheimer (1959). Freudenthal compared giving individual examples to the planting of seeds; each seed is planted with the expectation that it will flower. Wertheimer explained:

> Sensible generalization may come from one example viewing it structurally leading to reasonable understanding and to genuine discovery of essential features. The steps do not necessarily come into existence by viewing many cases and stating what is factually common in them. (p. 61)

## Johannot's Research on the Mathematical Reasoning of the Adolescent

Johannot (1974) presented an analysis of the mathematical reasoning of 112 students from 12 to 18 years of age representing the strong and weak students of their respective classes. Piaget (cited in Johannot, 1974) credited Johannot with having provided evidence of "the continuity between the child's reactions to those of adolescents when the latter reason on planes too abstract or too new for them" (p. 1). Johannot applied Piagetian theory to the classroom with tasks directly related to the curriculum. Piagetian constructs such as closure and reversibility were used by Johannot to interpret the learning of algebraic and arithmetic content by adolescents. Johannot demonstrated that the mathematical reasoning of the adolescent relative to the composition of operations (such as in simplifying complex fractions and solving equations) began with a stage in which the methods employed appeared to be very nearly those of "child prelogic" in which the adolescent may demonstrate proficiency in calculating, but is deficient in terms of Piaget's logical operations.

## Rote Versus Meaningful Understanding

Johannot (1974) found that in general, students' errors did not arise from a lack of knowledge of algebraic techniques. On the contrary, the students demonstrated a good understanding of "how" to perform algebraic operations.

Instead, students' errors originated from a lack of logical comprehension of the operations. Johannot stated the following "fundamental" law: "An algebraic or arithmetical rule will never be utilized correctly and its use cannot be generalized as long as it isn't preceded by a logical construction permitting realization of the automatic operation that it necessitates" (p. 53). As a corollary to this law, Johannot posited the need for logical comprehension to precede the automatic application of an arithmetic or algebraic operation. Although Johannot acknowledged that students can be taught how to perform mathematical operations before developing a logical structure for the operations, he asserted that knowing how should instead be based on the students' logical structure for the operations. Johannot's view of the importance of (meaningful) understanding is in close harmony with the view of Brownell, as can be seen in the following statement.

> For the student who doesn't see the sense of the rules and of the symbols that he uses mathematics must become an extremely difficult exercise obeying obscure laws, having no practical goal and obliging extra-ordinary efforts of memory since failing to logically relate one rule to another, he must learn them by heart and even, which is rather more serious, he will have to know each type of problem necessitating their use. He will be incapable of differentiating the fundamental form and will not recognize for example in the expression:
>
> $$4x^2y^2 - (x^2 + y^2 - z^2)$$
>
> a difference of two squares of the same form as $a^2 - b^2$. (Johannot, 1974, pp. 42-43)

Johannot outlined the evolution of students' lack of confidence in rotely learned algebraic solutions to problems. In attempting to assimilate new algebraic procedures into existing structures the student compares the task with any intuitive response, even approximate, which comes to mind in parallel with or prior to the algebraic solution. The incongruence between the new operation and his or her intuitive interpretations of existing operations (for example, between operations with whole numbers and rational numbers) forces the student to learn algebraic calculations on purely an abstract plane. The following typical student errors provided by Johannot illustrate this incongruence: "$\frac{1}{3} + \frac{2}{5} = \frac{1 + 2}{3 + 5} = \frac{3}{8}$, $\sqrt{7} - \sqrt{3} = \sqrt{7-3} = \sqrt{4} = 2$, and $3^2 + 3^5 = 3^7$" (p. 45).

Johannot's comments on the development of a lack of confidence are of interest in the present study. Successful students have demonstrated their ability to apply appropriately recognized operations. Does instrumental understanding imply confidence? How do the students' applications of intuitive false steps in an algorithm affect their application of the correct steps in the algorithm? Johannot (1974) suggested that intuition of a generalization made prior to an algebraic calculation gives it greater influence than that of the proper algebraic calculation. When do students apply false-intuitive generalizations after the algebraic generalization has been learned?

## Reversibility of Mental Operations

Piaget clearly distinguishes between two forms of reversibility: negation (or inversion) and compensation (or reciprocity). Essentially negation directly annuls an operation; in effect, it reverses the operation by "undoing" it. Compensation, on the other hand, refers to the reciprocal strategy of reversing an operation by leaving the operation alone and cancelling its effect. For example, in Piaget's balance beam experiment a weight is placed on one side of a balance at equilibrium. Equilibrium can be restored either by removing the weight just added (negation) or by adding an equivalent weight to the other side of the balance (compensation). At the stage of formal-operational thinking the adolescent has both strategies available (Collis, 1975a; Collis, 1975b; Adi, 1978).

Johannot (1974) found, for example, that the great majority of adolescents responded correctly to the question, "How many 2/3 are there in 1 whole?", but were unable to relate their responses to the problem, "How much does 1 divided by 2/3 make?" He noted that division will only be assimilated at the moment it becomes the inverse of multiplication. Otherwise both multiplication and division of fractions would need to be accommodated into their own individual structures, hindering the transfer of each operation to new situations.

Johannot (1974) stated that "reversibility in its two complementary forms is the condition of prime importance for logical and fruitful mathematical reasoning" (p. 88). He suggested that an operation such as addition is not complete for the adolescent until it is linked with the inverse operation of subtraction. Many elementary school mathematics textbooks attempt to form this link through the use of missing addends tasks, such as $5 + \square = 7$. A child's understanding of a given addition task, such as 5 + 2, is not complete, according to Johannot, until the child can demonstrate both forms of reversibility of the operation of addition. This would be manifested both as compensation (reciprocity), whereby the child determines what needs to be added to the 5 to obtain 7, and by negation, whereby the child solves the problem $\square = 7 - 5$. In compensation the child balances the 5 and the 7 by adding to the 5 until the sets are equivalent. In negation the child removes 5 from each set to determine the amount needed to balance the equation. Of importance to the present study is the appropriateness of extending these tasks to the study of algebra. For example, the task, "What polynomial added to $5x^2 + 3xy$ yields $3x^2 + y^2$?" may be solved by either compensation (reciprocity) or negation.

## Skemp's Theory of the Nature of Understanding

The term "understanding," as described by Skemp (1976), has two meanings. Mellin-Olsen (cited in Skemp, 1976) distinguished these two meanings as relational understanding ("knowing both what to do and why") and instrumental understanding (knowing "rules without reason"). Skemp (1976) noted, "For many pupils and their teachers the possession of ... a rule, and the ability to use it, [is] what [is] meant by 'understanding'" (p. 20).

## Instrumental Understanding

Herscovics (1979) restated Skemp's definition of instrumental understanding ("rules without reason") as "the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works" (p. 103). Kieran (1979) gave an example of instrumental understanding as "the child who can manipulate symbols but who lacks any awareness of the underlying ideas or relationships" (p. 24). Instrumental understanding may be further exemplified by Wertheimer's tale of the mechanical nature with which some students appeared to solve linear equations: "Like a machine that carried out without understanding a series of steps which resulted in an answer, so did they blindly repeat what they had learned by rote" (quoted by Luchins & Luchins, 1970, pp. 368-369). Collis (1975a) also pointed out the limited nature of many students' understanding of solving linear equations by noting, "Many students are able to give quite acceptable performances in the classroom without being aware of the notion of operating on operations as such but when asked to perform an analogous task with unusual operations find themselves unable to succeed" (p. 91).

Instrumental understanding, according to Skemp (1976), usually involves multiple rules. If all that is wanted is to be able to solve a particular kind of equation quickly and efficiently, then instrumental understanding is adequate. However, Skemp (1971) cautioned that the ability to apply meaningless rules is not sufficient to adapt one's knowledge to new situations. Skemp (cited by Harrison, 1967) further doubted that a student with only an instrumental understanding of mathematics would be able to solve problems efficiently. Skemp stated that the student with purely instrumental understanding "will continue to seek models for every problem encountered, but this will no longer do as new problems will arise that require intelligent modifications and combinations of existing concepts and processes, which is only possible if these have been [relationally] understood" (cited by Harrison, 1967, p. 105).

Many people view the products of mathematics (the answers to mathematical exercises) as mathematics. For these people, learning mathematics is learning to obtain these products and is most easily done instrumentally. The following account of one of Wertheimer's seminars suggests the position of the members of this product-oriented class with regard to the role of instrumental understanding:

> One seminar member said that when he had taken a course in college algebra all he had learned to do was to calculate or to use certain rules or procedures for certain types of problems. Neither he nor any other students in his college course

> seemed to be perturbed about not understanding algebra or its methods; as long as a procedure worked they were content .... Another seminar member asserted that one can only learn the behavior or responses that mathematicians use to solve a problem. It is not possible to understand their thought processes because they are private, subjective. As long as we learnt to do what they do and get solutions, we do not have to worry about understanding the nature of the process. He referred to Thorndike's Psychology of Algebra and said that Thorndike described how to inculcate in a student the responses which mathematicians use to solve certain kinds of algebra problems. Would we be able to have mass education in which algebra is taught to everybody if we did not have the rote learning method devised by Thorndike? (Luchins & Luchins, 1970, pp. 376-377)

For these seminar members and many people in general, understanding algebra is synonomous with knowing how to get the answers mathematicians do without necessarily knowing why.

Some teachers of algebra may focus on its symbols, as opposed to its concepts, because of the relative ease of such instruction. Bell (1976) pointed out, "Symbols are visible and movable but concepts are invisible and abstract; hence the tendency to teach symbol-transformations by rote and thus to detach the symbols from their meanings" (Chapter 2, p. 6). Wertheimer's (1959) example of a classroom lesson on the area of the parallelogram illustrates that some teachers and students view instrumental understanding as that understanding appropriate for the schools to teach. Wertheimer found that not only were students who were taught instrumentally to find the area of a parallelogram able to repeat word for word what the teacher had said, but there was transfer to all parallelograms pictured exactly the same way. To the students and their teacher this was evidence that the lesson was "understood." The inability of the majority of the class to find the area of a parallelogram not pictured like those in class was viewed as reasonable by the students since they had not learned how to calculate areas for the "new" kind of figure. Even the teacher viewed this task as a "trick question."

How do students who have demonstrated an instrumental understanding of mathematics view mathematics? Collis (1975a) reported:

> These students often appeared to have tacitly assumed that mathematics per se was not meaningful and was not really related to reality. It was, instead, a multiplicity of abstract and often conflicting rules and procedures from which one was expected to make an appropriate selection on being given a particular cue. (p. 11)

For students with an instrumental understanding of mathematics, the learning of more and more mathematics increases the difficulty of distinguishing between the growing number of apparently disjoint rules and procedures. Since they understand the rules as disjoint bits of knowledge, they view mathematics as the collection of these disjoint bits of knowledge.

## Relational Understanding

Herscovics (1979) expanded on Skemp's definition of relational understanding ("knowing what to do and why") and defined relational understanding as "the ability to deduce specific rules or procedures from more general mathematical relationships" (p. 103). Kieran (1979) extended this definition slightly "to include the ability to establish relationships between concepts and to make connections" (p. 24). So described, relational understanding includes the ability to grasp the structure of a subject "in a way that permits many other things to be related to it meaningfully" (Bruner, 1960, p. 7). Relational understanding includes knowing how things are related.

Other components of relational understanding may be inferred from Skemp's earlier writing. As Harrison (1967) said:

> Mathematical thinking is particularly characterized by the process of exploration and generalization that derives new class-concepts and operations from existing ones and that applies class-concepts and operations in fields different from those of their origin. (p. 117)

Hence relational understanding includes the ability to generalize new operations and concepts and the ability to transfer mathematical skills and concepts. The degree to which a student's written work corresponds to a learned algorithm is insufficient evidence of the type of understanding that the student possesses. In order to determine the meaningfulness of a student's response one must probe deeper into the nature of the student's mental operations.

Harrison (1967) expanded on the reflective nature of (relational) understanding. Not only should relational understanding at its height include the necessary concepts and operations, but its operations should be amenable to choice and modification in the light of information gained from previous attempts at solving a problem. Relational understanding, according to Harrison, can be implied by the learner's use of information from unsuccessful, as well as successful, attempts to solve problems.

Skemp (1976) identified some affective characteristics associated with relational understanding. Discovering the structure within a given area of knowledge becomes an intrinsically satisfying goal in itself and leads to greater confidence in one's ability to find flexible approaches to solving problems without help. According to Skemp, relational understanding is both self-continuing and self-rewarding.

The sensible performance of operations was considered by Wertheimer (1959) as a natural characteristic of (relational) understanding. It is practical to make some operations habitual. In doing so, the mind may be freed to deal with higher problems involved in a task.

The spiral nature of mathematics makes it advantageous to be able to apply previously learned processes efficiently and accurately in the learning of more complicated processes. Skemp (1971) stated

> If we are to make progress in mathematics it is, indeed, essential that the elementary processes become automatic, thus freeing our attention to concentrate on the new ideas which are being learnt -- which in turn must become automatic....

> In mathematics, this is done by detaching the symbols from their concepts, and manipulating them according to well-formed habits without attention to their meaning. This automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols. (pp. 88-89)

Relational understanding of previously learned processes includes the ability to apply at will either knowledge of "how" or knowledge of "why" to new situations.

In the present study no distinction was made between the terms "instrumental" and "rote." The goal of the connectionist learning theory is the development of instrumental understanding. Similarly, no distinction was made between "meaningful" and "relational." The goal of the meaningful learning theory is the development of relational understanding. Those comments made with regard to rote learning and meaningful learning apply correspondingly to instrumental understanding and relational understanding.

## Questions of Interest

Given the foregoing theoretical discussion, observations were made for the following questions of main interest:

Question 1. What evidence of a reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

Question 2. What evidence of a flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

The following questions are of secondary interest in the present investigation and will be addressed in less detail:

Question 5. What evidence of the use of false transfers is provided by someone who is successful in a basic algebra course?

Question 6. What evidence of the role of perception and memory on the solution of algebraic problems is provided by someone who is successful in a basic algebra course?

Question 7. What evidence does someone who is successful in a basic algebra course provide of his attitude towards algebra and his perception of his understanding of algebra?

# CHAPTER II

## REVIEW OF THE LITERATURE

### Introduction

Weaver and Suydam (1972) pointed out the relative scarcity of studies of meaningful learning involving the secondary school curriculum. Studies of secondary school mathematics have tended to compare various instructional techniques as opposed to describing the processes of the learner. Those studies that have focused on the learner were generally studies of problem-solving processes or of student errors as measures of understanding, as opposed to studies of abilities related to algebraic skills and understanding.

Few studies conducted before 1970 focused on students' understanding of the processes and operations of algebra. Breslich (1939) used high school juniors' and seniors' verbal responses to analyze the errors they made in applying many of the algorithms of first-year algebra. Breslich's analysis was limited by the fact that, as problems became routine for the students, the steps became fused so completely that introspective analysis was difficult. Thus in the case of $-25xy \div (+5x)$, the students commonly omitted mention of the process by which they arrived at the quotient (only 8 of the 52 students who solved it correctly mentioned the process while solving the problem). Breslich observed that "when a process takes on the form of a routine procedure, it is often difficult to determine whether or not the pupil ever understood it" (p. 85). Like an adult asked to think aloud while finding the sum of 3 + 4, the students would immediately provide solutions without providing any indication of their solution paths. To study students' solution processes it is necessary to probe their thinking so that adequate traces of their solution paths will be available for analysis.

Shevarev (1975) investigated student errors in learning algebra by observing the students in class and analyzing their notebooks. As contrasted with Breslich, who studied errors due to not knowing a rule or not knowing how to apply a known rule, Shevarev limited his investigation to "cases where a pupil knows a rule, is able to apply it, but nevertheless acts contrary to it" (p. 1). In harmony with Thorndike's (1926) conception of learning algebraic computation as the formation and organization of a hierarchy of connections or bonds, Shevarev described the learning of algebraic skills as an evolution from rules to unconscious connections through the medium of practice. Errors in which a student knows a rule, is able to apply it, but does not apply it correctly were viewed by Shevarev as the results of incorrect connections.

Johannot (1974) showed that the adolescent working on algebraic tasks operates in much the same way as a child does on arithmetic tasks. Others (for example, Thorndike, 1926; Judd, 1939) had suggested the appropriateness of extending the research implications from children's learning of arithmetic to the adolescent's learning of algebra, but Johannot demonstrated the feasibility and value of such an extension. Johannot found that tasks relevant to measuring children's understanding of arithmetic may in a similar way be relevant in measuring the adolescent's understanding of algebra.

Perhaps most notable among the pre-1970 studies of students' algebraic abilities were those included in the investigations of mathematical abilities conducted by Krutetskii (1963, 1969, 1973, 1976). Krutetskii's research has been discussed by several mathematics educators (see, for example, Hatfield, 1979; Kantowski, 1979; Kilpatrick, 1978; Lester, 1980; Rachlin, 1979). For the present investigation Krutetskii's study of abilities provided a clarification of research directions for investigating students' ability to generalize algebraic processes and operations, their flexibility of approach to solving algebraic problems, their ability to reverse algebraic operations, and their ability to transfer algebraic processes and operations to new fields.

Recently there has been growing interest in investigating the processes involved in the learning of algebra. These studies have emerged largely from three different sources: information-processing theory (Carry, Lewis, & Bernard, 1980; Davis, 1975a, 1975b; Davis & McKnight, 1976, 1979; Matz, 1979), Piagetian theory of cognitive development (Collis, 1975a, 1975b; Firth, 1975; Kuchemann, 1978; Wagner, 1977), and Vygotskiian theory of cognitive development (Kieran, 1979). These studies, although fundamentally different, possess the common aim of examining the processes involved in the learning of algebra. Further, in each of the studies a theory is developed or confirmed on the basis of an analysis of students' errors in the perception of algebraic symbolism and in the performance of algebraic skills.

## Reversibility and Flexibility

### The Ability to Reverse Mental Operations

Johannot (1974) identified three stages of reversibility based on adolescents' performance on the tasks: "How many 2/3 are in 1 whole?" and "What does $1/\frac{1}{2}$ represent?" The stages range from complete irreversibility to complete reversibiity. In the stage of complete irreversibility, the adolescent was characterized by a complete incomprehension of the notation for complex fractions resulting in the erroneous use of rules and a lack of comprehension for the meaning of applied rules. This stage was followed by one in which the adolescent demonstrated some initial comprehension of the notation for complex fractions as evidenced by variable success with the application of rules and a continuing incomprehension of their meaning. Only during the final stage did the adolescents demonstrate comprehension of the meaning of the rules. This stage was also marked by the correct application of the rules for simplifying complex fractions and a complete comprehension of the notation for complex fractions.

Collis (1975a, 1975b) demonstrated the evolution of three levels of reversibility in terms of finding the value of "y" in the equation "$y + 4 = 7$." In contrast to Johannot's levels, which were based on the student's developing comprehension of a reversible operation, Collis's levels ordered the development of a student's flexible application of both negation and compensation. At the first level (8 to 9 years) an early form of compensation is applied in counting from "4" until "7" is reached. The child balances a set of four objects with a set of 7 objects by adding on to the first set. At the second level (10 to 15 years) the child regards "y" as a unique number like "7" but for the time being unknown. The equation is solved by "undoing" (negating) the effect of adding "4" by subtracting it. And since "$y + 4$" is the same as "7," the negation is accomplished by subtracting "4" from "7." In the final level (16+ years) the adolescent negates each operation without upsetting the existing state of equality and readily allows for the possibility of returning to the original statement: $y + 4 + (-4) = 7 + (-4)$.

Collis (1975a, 1975b) suggested that simple compensation occurs first in children's solutions of missing addends tasks. Does it also occur first in adolescents' solutions of polynomial versions of the missing addends tasks? How flexible are adolescents in their use of negation and compensation techniques when working with algebraic polynomials? How does this flexibility compare to their flexibility of approach (negation and compensation) to operations involving radical expressions and complex numbers?

Krutetskii (1969, 1976) found that students who applied a short-form algorithm for multiplying the sum and difference of two terms varied in their ability to recognize and apply successfully the reverse operation of factoring a difference of two square terms. Capable pupils were able to reverse the direction of their thought associations "on the spot" so that the formation of reverse associations was almost simultaneous with the formation of "direct" associations. Average pupils required hints of varying degrees of specificity and would at some point "catch on." After that, they would move quickly through the exercises, the "reverse" associations having been established. Less capable pupils had difficulty in establishing "direct" and "reverse" associations. Many hints (in the form of instruction) and many exercises were required to form each one-way association. Krutetskii (1969) found that "many of the less capable students could not relate 'reverse' associations with their corresponding 'direct' associations" (p. 51).

Shapiro's (cited in Krutetskii, 1976) research provided several examples of the variation in students' ability to reverse mental operations. One such example is as follows:

> When the concept of a complex number was introduced in grade ten, the pupils were asked to multiply $(a + bi)(a - bi)$. Obtaining $a^2 + b^2$ as an answer, a capable pupil, D, at once noticed (a reverse association was "set in motion") that "now, it turns out, the sum of squares is factored. Before it was impossible, but now it is possible, since we have learned about imaginary numbers." And there and then the pupil posed the question of factoring the sum of any even powers (such as $a^4 + b^4 = (a^2 + b^2i)(a^2 - b^2i)$. P, an average pupil, did the multiplication correctly but did not notice anything. When he was asked ten minutes later to decompose the expression $x^2 + y^2$, he was very surprised and declared that it could not be done. (p. 291)

Johannot's and Collis's discussions of reversibility were limited to work with single unknown terms. Krutetskii extended the discussion of reversibility to multiplication tasks with two unknowns. In the expression $a + b = c$, the reversibility of addition incorporates three variations: where c is known and either a or b is given and where c is given and neither a nor b is given. In the present study the view is taken that for children to possess complete reversibility of addition of whole numbers they should be able to solve problems involving all three variations: $\square + 2 = 7$, $5 + \triangle = 7$, and $\square + \triangle = 7$. This view incorporates and extends the discussions of reversibility of

Johannot, Collis, and Krutetskii and is readily transferred to the operations of algebra in the present investigation.

An adolescent who possesses complete reversibility of the addition of $5x^2 + 3xy$ and $2x^2 - 3xy + y^2$ should be able to solve the following three tasks:

(1) What polynomial added to $5x^2 + 3xy$ yields $3x^2 + y^2$?

(2) The trinomial $2x^2 - 3xy + y^2$ added to what polynomial yields $3x^2 + y^2$?

(3) Find two polynomials such that their sum is $3x^2 + y^2$.

The last of these tasks is more complicated than the first two. In this case the solutions are not unique. Negation and compensation can be applied with some additional complexity. Negation can be applied by choosing a polynomial and then subtracting it from $3x^2 + y^2$. Compensation can be applied by searching for two appropriate polynomials term-by-term. With multiplication of polynomials, compensation is known as factoring and negation is known as division. For example, the common method for finding two polynomials whose product is $x^2 + 5x + 6$ is to search for the two polynomials via compensation. In the solution of problems involving multistep operations, such as addition of fractions, compensation may necessitate the recognition of intermediate steps such as reducing to lowest terms.

### Flexibility of Mathematical Approach

Krutetskii (1976) found that students varied in the flexibility of their approach to mathematical material. Capable students were characterized by the ability to switch without any difficulty from one method of approach to another. A previously stereotyped approach had no binding effect on their solution to a problem. Average students had a much harder time switching to a new method on a problem that had already been solved. Their thoughts returned every now and then to the pattern already found. The less capable students, as characterized by Krutetskii (1976),

> are marked by inertness, slugishness, and constraint in their thinking in the realm of mathematical relations and operations, by the settled, stereotyped character of their operations, and by the importunate retention in their minds of a previous principle of solution or a method of operation that exerts an inhibiting influence when an operation needs to be reconstructed, all of which determine the pronounced difficulties they have in switching from one mental operation to a qualitatively different one. (pp. 282-283)

## Generalization and Transfer

### Generalization and Transfer of the Processes and Operations for Algebraic Expressions

Krutetskii's investigations of the ability to generalize involved 120 pupils in four stages. There were 26 series of tests. Three of the tests from these series are of direct relevance to the present study. The first two were used to measure the students' ability to extend a class of algebraic skills. The third was intended to measure the students' ability to generalize to phenomena not yet encountered.

The first test (Krutetskii, 1969, 1976) was designed to classify how students who were taught the formula for squaring a binomial differentiated instances when the formula could or could not be applied and applied it when appropriate. The test consisted of a "staircase" of increasingly more complex generalizations of a principle and a corresponding "staircase" of increasingly more complex instances in which the principle did not apply. The "staircase" of eight problems for which the principle was appropriate ranged in difficulty from "$(a + b)^2 =$" to "$(C + D + E)(E + C + D) =$". Krutetskii (1969) described how the ability to generalize was to be inferred from the verbal traces of a student's solutions:

> We judged the development of the ability to generalize by observing how well a pupil was able to isolate general information which appeared in various problems, how well he made the transition from solving simple problems to solving more complex ones of the same type, and how well he was able to differentiate problems of one type from problems of another type that were similar in appearance. (p. 19)

Krutetskii(1969) noted that the first test was inadequate for characterizing the ability to generalize of students with "little capacity" for mathematics. The students could not do even the simplest variants correctly without a great deal of guidance. Krutetskii created a second test for use when the first test was too difficult for students. In this test increasingly more complicated problems on the application of a formula for finding the square of a binomial were gradually introduced. Seven variations were established by using different letters, introducing coefficients, introducing exponents, and using combinations of these. The initial problem was "$(a + b)^2 =$ " and the terminal problem was "$(\frac{1}{4}x^n + 0.2y^{b+1})^2 =$". This second test was used to study the ease with which a student passed from solving simple problems to solving more complex ones of the same type.

The third test was designed to clarify the degree to which a student could produce a generalization concerning a number of similar objects after analyzing just one object. In this test students were given a problem that they might or might not solve and were asked to create one "like" the given one. For example, students were given the task "Think of a trinomial that could be factored by putting the term $2a^2b$ in parentheses." After becoming familiar with the problem (which could, but did not necessarily, imply solving it), the students were to create a problem of the same type. Krutetskii based his interpretation of a student's ability to generalize from a single instance on how the student "analyzes the structure of a problem (or the structure of a problem's solution) and how he reasons, isolating the relevant and irrelevant from the standpoint of type" (p. 126). Performance on the task was categorized by the degree to which the similar problems generated were generalizations of the problems given, the increased complexity of the task, and the semantic and syntactic focus of the change.

Krutetskii distinguished the characteristics of ability in terms of the level of mathematical capability of the students. The more capable the pupils, the more rapidly and broadly they tended to generalize mathematical materials. At its maximum this ability was characterized by "on the spot" generalization.

The average students, according to Krutetskii, did not make "on the spot" generalizations that extended to all possible variants and combinations. But they were able to approach such generalizations by being provided increasingly more complex exercises in which the unessential features would vary (as is commonly done in textbooks).

Krutetskii (1976) expanded on the characteristics of the "less" capable pupils, "It was always very hard for our [less-capable] pupils to abstract themselves from concrete numerical expressions. Their basic difficulties in the study of algebra were related to the necessity of thinking of numbers in terms of generalized letter designations" (p. 253). Sounding like a frustrated teacher, Krutetskii lamented, "One would think that a pupil has understood and generalized the solution of an algebra problem. But while solving the next problem, he would experience the same difficulties and would need the experimenter's help" (p. 253).

Krutetskii's analysis of mathematical abilities was based on the differences in performance between extremely gifted students and average and less capable students. As a result, Krutetskii investigated the presence in average and below average students of the processes that he found in the gifted students.

Krutetskii did not investigate, for example, students' false generalizations. He did not look at the error patterns of gifted students on difficult tasks and investigate the presence of these patterns in average and below average students. Nor did he examine the false generalizations made by average and below average students and investigate the presence of these false generalizations in the responses of the gifted students. The literature on false generalizations and false transfers (the generalization of false generalizations to other topics) of algebra are discussed in the next section.

## False Generalization and Transfer

Johannot (1974) found that in general, his students' errors did not arise from a lack of knowledge of algebraic techniques. On the contrary, the students demonstrated a good understanding of "how" to perform algebraic operations. In attempting to assimilate new algebraic procedures into existing structures the student compares the task with any intuitive response, even approximate, which comes to mind at the same time as or prior to the algebraic solution. The incongruence between the new operation and his or her intuitive interpretations of existing operations (for example, between operations with whole numbers and rational numbers) forces the student to learn algebraic calculations on a purely abstract plane, without any connection with previously learned operations.

Johannot suggested that intuition of a generalization made prior to an algebraic calculation gives it greater influence than that of the proper algebraic calculation. Matz (1979) elaborated on this point in her theory of errors: "When solving problems, a student tries to fit new situations to rules he has acquired from prior experience in arithmetic and algebra. If he finds such a fit, the rule tells him how to proceed. If he has no rule that fits, he may generalize the rule, sometimes violating the semantics of algebra" (p. 2). Carry, Lewis, and Bernard (1980) also found this form of error so frequently that they suggested the possibility of "discouraging students from trying to figure out algebraic operations that they don't understand" (p. 121)! When do students apply false intuitive generalizations after an algebraic generalization has been learned? More information on the processes that lie beneath the successful student's use of an algorithm may tell us much about the appropriateness of our instruction.

Wertheimer (1959) and Shevarev (1975) have pointed out the importance of investigating the role of these false generalizations in explaining students' understanding of algebra. How stable are false generalizations? How is their occurrence related to the level of difficulty of a given task?

## Generalization and Transfer of the Processes and Operations of Algebraic Equation

The investigation of reversibility of algebraic operations bridges the void between the mental processes for algebraic operations and the solution of algebraic equations. A statement of the form a * b = c, where a and c are given polynomials and * is a given operation, is an equation. Matz' (1979) theory of high school algebra errors provides a description of the type of errors that might develop from such a dual use of the equal sign. Matz claimed that "a student's errors can be viewed as the surface manifestations of underlying bugs in his processes for fitting his current knowledge to a new situation" (p. 3). Two major locations for errors to develop in the learning of algebra would thus be at the transition from arithmetic usage to algebraic usage of symbols and within algebra from the notation used in operations with algebraic expressions to the notation used in algebraic equations. At both these locations several concepts of algebra should take on new meanings for the students. In the elementary school "3 x 4" is thought of as a problem or a question; like the next-to-last note of a melody, it cries out for completion. In the high school 3i indicates both a process, a number, and perhaps an answer to some problem. Other examples of algebraic expressions which may have varied meanings are $\sqrt{2}$, $3 + 2i$, and $\sqrt{3} + \sqrt{2}$ (Davis, 1975).

In the learning of algebra another shift in the use of concepts occurs when moving from the topic of algebraic operations to the topic of algebraic equations. Duality appears this time in the use of the equals sign. In working with algebraic operations the equals sign is usually used to herald the answer (or intermediate steps toward the answer). A student generally moves from left to right, from task to solution. In solving equations, the equal sign serves dual purposes. It indicates the equality of two expressions on which the student performs some joint operations, while at the same time some operations are still carried out in single members, perhaps even in the right-hand member of the equation (Kieran 1979; Matz, 1979).

Students are normally taught to apply negation to solve algebraic equations. Alternative approaches which train the students to apply compensation in equation solving have been suggested by Adi (1978) and Kieran (1979). Perhaps the dual use of the equal sign could more meaningfully be developed by including in the students' training, experiences in which the students use compensation as well as negation to solve equations.

## Related Abilities and Concepts

### Perception and Memory

Perception of a problem plays a large role in determining which operations are appropriate for a solution. Krutetskii (1976) investigated the ability of students to "grasp" the structure of a problem: to analyze those aspects of the problem that have a basic mathematical meaning, that are idiosyncratic to the given problem, and that are inessential for its solution. Krutetskii described the nature of the capable student's marked ability to perceive immediately not only the individual elements of a structure, but also distinctive "mathematically meaningful structures" as a "composite whole." As Davis (1975) has observed, the capable student perceives a problem in meaningful "chunks" that can be held and operated on in short term memory. This ability to "grasp" the structure of a problem is later manifested in the "on the spot" abilities observed in the student's processing of information.

Krutetskii (1976) stated that the average student tended to perceive a problem by its separate mathematical elements. "The average pupil must face the special task of connecting the mathematical elements of a problem, and in the process of analysis and synthesis he finds this connection" (p. 228).

The less capable students, according to Menchinskaya (cited in Krutetskii, 1976), had great difficulty abstracting themselves from the concrete content of a problem. "Beyond the subject of the problem, they do not see its real mathematical meaning" (p. 228). Krutetskii (1976) gave less capable students the task of factoring the expression $27x^3 + 27x^2 + 9x + 1$:

> They were not able to perceive this expression as the expansion of the cube of a sum of two numbers, because they looked at each term in isolation, without any connection with the others. If some of them succeeded in solving such a problem correctly, they did it "illegally", guided only by the first and the last terms (this was easily discovered when, after the solution, this example was given:
>
> $$8y^3 + 8y^2 + 8y + 1$$
>
> -all of them solved it using the same formula). (p. 229)

Kilpatrick (1978) described the variations in Krutetskii's subjects' memory for mathematical problems that the subjects had solved. The capable students tended to remember problems in terms of their mathematical structure. The less-capable student tended to remember problems in terms of the context of a particular problem (if at all). The tendency for capable students to remember the semantic aspects of problems and for less capable students to remember the syntactic aspects of a problem was also noted by Silver (1978) Krutetskii (1976) characterized the average student as attempting to remember both the semantic and the syntactic components of a problem. The average student's ability to remember problems over time was adversely affected by the "overburden with surplus information" (p. 297).

Shevarev (1975) identified three types of errors students make because of an incorrect perception of a task:

(1) Having prior knowledge, the pupil often recognizes only certain features of the algebraic expression given him, and omits equally essential ones. For example, in simplifying algebraic expressions like $a^m \cdot a^n$ the student focuses on the "general" features of the expression: the absence of plus and minus signs in the base and the presence of two exponents. The student learns to solve a large number of problems with these features. Later the student is given problems of the form $(a^m)^n$ with the same "general" features. Although the "specific" features of the expressions differ (the first expression has two identical bases, while the second expression has only one base), the student has a tendency to apply the incorrect connection $(a^m)^n = a^{m+n}$.

(2) Rules are not generalized sufficiently but contain features that are not essential; i.e., the student perceives the specific features as being part of a rule and does not generalize the rule sufficiently. For example, Shevarev found that students viewed rules such as $\frac{a^m}{a^n} = a^{m-n}$ as applying only in situations when "a" was a single letter. Generalizing the rule to include situations when "a" was a polynomial tended to fade over time. Each time the student correctly solved a problem involving division of monomials, the connection that this applies only to single letter factors was reinforced.

(3) The student does not perceive all features due to a focus on particular, highly visible features. For example, in multiplying $(a\sqrt{b} - b^2\sqrt{a})^2$ the $\sqrt{b}$ dominated several students' perception of the problem. The common error $ab - 2ab^2\sqrt{ab} + b^2a$ was made by these students. In most cases where an error occurred the students had demonstrated that they had no difficulty in squaring other binomials.

The three types of incorrect connections identified by Shevarev emphasize the role of perception in the errors made by algebra students. Errors in the application of a rule were initiated by perceptions that were too general, too specific, or too focused. The role of perception is closely tied to the role of memory in the errors algebra students make. Shevarev pointed out that both incorrect and correct bonds are stored in memory. Of importance to this study is how perception and memory are tied to the solution of algebraic problems. Is the incorrect connection more likely to appear if a student focuses on a particular feature of the problem (for example, that it involves complex numbers)?

## Meaning of Variable

The symbols of algebra take on a variety of meanings depending on their use in particular situations. Consider, for example, the use of the symbol x in the following: $2x + 3$, $2x + 3 = 5$, $3x^2$, $3/2\ x$, $(x + 3)^2 = x^2 + 6x + 9$, $(x + 3)^2 = x^2 + 2x + 6$ (Van Engen, 1953). With all the variety of syntactic interference it is no surprise that the meaning of variables (or the lack of meaning) is a source of difficulty. But what is surprising is the extent and depth of the problem. Consider the following task from Firth's (1975, p. 41) study of 15 year olds' concept of variable:

x is any number
(a) Write the number which is 3 more than x.
(b) Write the number which is 5 less than x.
(c) Write the number which is twice as big as x.
(d) Write the number which is 50% bigger than x.

Firth found that of the 17 students given the task only 10, 11, 9, and 7 students, respectively, solved the parts correctly. Firth noted that 5 of the 17 students solved the entire task incorrectly by first choosing a value for x and using the value throughout the exercise. What was the cause of their error? Was it caused by a misunderstanding of the task or of the meaning of variable? Did the students confuse the statement "x is any number" with "choose a value for x"? Perhaps some follow-up questions would have provided a clearer picture.

Wagner (1977) confirmed the extent of the difficulty that students had with the concept of variable. Wagner found that the meanings that students associate with literal variables may be based on their place in the alphabet. Collis (1975b) confirmed this difficulty. In solving problems like determining when $a + b + c = a + b + d$ is true, the students Collis studied checked off the a and b in each member and were left with $c = d$. Most of these students (ages 13 to 15) then stated that c and d obviously can not be the same and hence gave "never true" as a response. Collis found that students had three levels of meaning for variables depending on what was "real" for them. The lowest level seemed to map a variable into a specific number that seemed appropriate at the time (as had Firth's students). If after one trial this value did not work, the student gave up on the task. At the second level the students were willing to try a whole set of numbers in a "guess and test" technique. Students with the highest level of meaning viewed variables as "generalized" numbers.

Küchemann (1978) built on Collis's research to refine and test the meaning of variable. Letters were found by Küchemann to have six separate connotations in elementary algebra: letter evaluated, letter ignored, letter as object, letter as specific unknown, letter as generalized number, and letter as variable. All six uses were included in the algebra examination designed by the Concepts in Secondary Mathematics and Science (CSMS) Project (Hart,1981). This examination was used in the present study to measure a student's understanding of variable. Items on the tests were created within a Piagetian framework and were individually classified in terms of Piaget's stages of development.

## Summary

Research on students' understanding of mathematics has traditionally been conducted at the elementary school level. Only within the last decade has there been much research into the processes of learning higher level mathematics. The literature on the understanding of algebra has largely developed from three theoretical sources: information processing theory, extensions of Piagetian theory of cognitive development, and extensions of Vygotskian theory of cognitive development. The review presented here was a compilation of the literature applicable to the study of reversibility, flexibility, and the ability to generalize and transfer algebraic processes and operations. The theoretical background and review of the literature served as the foundation for the research questions and research methodology described in the next chapter.

# CHAPTER III

# THE PLAN OF THE STUDY

An essential measure of the reliability of a study's findings is the degree to which the study is replicable. One criticism of the Soviet style of reporting research is that "adequate" descriptions of the methods of carrying out the study are rarely provided. In order to be adequate the description of a research study must include a discussion of the instruments, subjects, and procedures in sufficient detail to permit replication.

Four main questions will be addressed in this chapter: Why? What? How? To whom? The answers to these questions provide a description of: why a Soviet "ascertaining" methodology was chosen for use in the present study; what instruments were used, how they were developed, and how they were used in the analysis of four processes of understanding; how the study was carried out; and finally, who the subjects were and how they were selected.

## Why? A Rationale for the Choice of Methodology

The present investigation sought to uncover information about and gain a greater insight into the internalized operations used by successful students in solving algebra problems. The investigation of four processes of understanding was designed to go beyond product measures of students' learning to their internalized thought processes. With all its inherent limitations, a clinical approach has been recognized by mathematics educators as the most appropriate methodology for investigating internalized operations of thought (see, for example, Hatfield, 1979; Kantowski, 1974; Kilpatrick, 1969; Lester, 1980).

Because the research questions in this study concern the algebra student's thought processes, a clinical approach is the appropriate methodology. The particular type of clinical study chosen for this investigation was the ascertaining experiment.

Kantowski (1979) described the ascertaining experiment as follows:

> As the name implies, the purpose of the ascertaining experiment is to examine a knowledge state or pattern of behavior as it exists in some population. It is most often undertaken to gather information about the status of a skill or ability or to determine error patterns as a prelude to diagnosing reasons for such errors. The clinical methodology is generally employed. Data are collected by having individual students think aloud while doing mathematics; at times students are also asked to examine their work in retrospect and to explain procedures used. Results are reported by noting generalizations formed on the basis of qualitative analysis (often professional judgements) of the data. (p. 133)

The ascertaining experiment was designed "to determine the peculiarities of assimilation and application of knowledge and of the formation of habits and modes of intellectual activity under given already formed conditions of learning" (Kalmykova, 1967, p. 14).

The present investigation was conducted as an ascertaining experiment for the following reasons:

1) The clinical interaction of the ascertaining experiment provides a vehicle for analyzing the internal operations used by students in the understanding of algebra. The use of the thinking aloud procedure mixed with retrospection through directed questioning provides traces of thought not available through paper-and-pencil means.

2) The study focused on algebraic abilities that were already in the process of forming. The flexible nature of the ascertaining experiment permits the interviewer to follow the flow of a student's thought with impromptu questioning. This is of particular value in tracing the source of false generalizations (see Question 5, p. 37) and self-perceptions and attitudes (see Question 7, p. 37).

3) A student's understanding of algebra is dynamic. It changes with time and experience. The long-term duration of the ascertaining experiment has been designed to capture the quality of the dynamic nature of learning. The use of and response to hints provides a sense of the potential for the development of abilities.

## What? The Instruments of Investigation

Various instruments were used to describe the sample and to answer the research questions. The students' responses on the instruments helped to clarify similarities and differences among the students. Most instruments were used to obtain traces of the student's solution paths on problems closely related to the content of basic algebra.

## Descriptive Instruments

Six instruments were used to provide descriptions of the subjects. The instruments provided information on a student's understanding of variable representation; ability to generalize from a given instance of a rule to increasingly more complicated instances of the rule; perception of similarity among tasks on operations with algebraic expressions; memory for algebraic tasks; preference for solving different types of algebraic tasks; and goals, background, and self-perception with regards to the learning of basic algebra.

### CSMS Algebra Test

An adaption of the algebra test developed by the Concepts in Secondary School Mathematics and Science Project (CSMS) was used to describe a student's understanding of algebraic symbolization (Hart, 1980). The CSMS Algebra Test is a half-hour paper-and-pencil test designed to measure adolescents' understanding of six different ways that letters are used as variables in algebra. The test was based on the research of Collis (1975a, 1975b) described in Chapter II. The test was adapted for use in the present study by "Americanizing" the vocabulary and units of measure. In addition to its use in this study as a descriptor of entering behavior, its value as a predictor of success in basic algebra was explored.

A subject's responses to the CSMS Algebra Test were used to describe her understanding of the use of variables. Error patterns on the test were used in analyzing data from interviews. The subject's responses were also compared to Küchemann's (1978) classification of items in terms of Piaget's stages of concrete and formal operational thought.

## Generalization Test

The Generalization Test (see Appendix C) was used to measure a student's ability to generalize from a known rule to increasingly more complex variants of the rule. The Generalization Test was similar to two tests used by Krutetskii (1969; 1976) in his investigation of mathematical abilities. (Krutetskii's tests were detailed in Chapter II.) The Generalization Test was developed in the two pilot studies described in Appendix A. In its final form the Generalization Test included four rules:

Rule 1: $(a^n)^m = a^{nm}$

Rule 2: $(A + B)^2 = A^2 + 2AB + B^2$

Rule 3: $(a, b) + (c, d) = (ad + bc, bd)$

Rule 4: $(a, b) \cdot (c, d) = (ac, bd)$

The student was requested to apply a given rule to a "staircase" of increasingly more complex instances. Each variant was designed to extend the class of instances for which the rule applied (much like the ordered series of exercises found in most algebra texts). Even if the student knew another way to perform the indicated operations, she was requested to apply first the given rule to each variant and then to check her work with the other method. For example, Rule 2 was to be applied to finding the square of binomials ranging from $(x + y)^2$ to $(x^{3m} + 1)^2$ before the student was permitted to use the FOIL method to check her work.

In the last series of tasks on the Generalization Test the student was provided both Rules 3 and 4 and was requested to apply them appropriately to mixed instances. In this series the student had to determine which rule to apply and then apply it correctly. Thus the Generalization Test provided evidence of the nature of a student's instrumental understanding of algebra -- her ability to apply rules in appropriately recognized situations. Since the students were selected largely on the basis of their instructor's classification of them as above average, they were expected to have little difficulty with this test. The errors that did occur were used in the analysis of the interview data.

The Generalization Test was completed by each student individually with the experimenter observing her procedure. After completing the test the student was asked to retrospect on two aspects of the test: her self-perception and attitudes with regards to the instrument and her perception of a relationship of Rule 3 and Rule 4 to anything she had done before in mathematics. Since the instrument was given during the fourth week of the course all of the students had recently completed or were currently working in a chapter dealing with operations with algebraic fractions. The information provided by this interview was used in answering the research questions of secondary interest.

### Classification Test

The Classification Test (see Appendix C) was used to describe the way students classified tasks they had encountered during the quarter. The items were taken from the appropriate chapters of the textbook and from the algebra interview tasks.

The Classification Test was based on instruments used by Silver (1978) and Kantowski (1977) in their investigations of junior high school students' problem-solving processes. The Classification Test was developed in the two pilot studies described in Appendix A. The test was altered from that used in the second pilot study by including only tasks that involve operations with algebraic expressions.

The test consisted of 21 algebra tasks selected to represent the four basic operations with polynomials, algebraic fractions, radical expressions, and complex numbers in "textbook" or "verbal-problem-to-find" forms. Table 3.1 gives the distribution of tasks in each category. The "textbook" tasks were selected from the students' textbook. These nine tasks were of the form A * B, where A and B were given algebraic expression and * was a given operation; for example, "Simplify: $(5x - 2)(2x + 1)$." The "verbal-problem-to-find" forms were drawn from the algebra interview tasks used in the study. The tasks were of the form A * B = C, where either A or B was to be determined with the other information given (seven tasks); both A and B were to be determined with C and the operation given (three tasks); or A,B, and C were to be determined with only the operation given (two tasks). These 12 tasks tended to be wordy and hence were named by one of the students "verbal-problem-to-find" exercises; for example, "Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$." The algebra interview tasks are described in greater detail later in the chapter.

Each task of the Classification Test was typed on a single 3-by-5-inch note card. In the right-hand corner of each card was a blank. On the first card the number "1" had been written in the blank and circled. The student was requested to make a pile of all those cards containing tasks like the one on the first card. The student was then asked to place a "1" in the blank on each card in the pile. Next, the student was to choose any of the remaining cards, place a "2" in the blank, and circle the "2." Then the student was requested to make a pile of all those remaining cards containing tasks like the one on the card with the circled two. The process was continued until all the cards were numbered. Only the first card in each pile had a circled number.

The Classification Test was completed by each student individually with the experimenter observing the procedure. As soon as the test was completed, the student was administered the Mathematical Memory Test. Then she was given back the cards and asked to reflect on her classification scheme.

The schemes were used to describe the students. Of interest were the number of piles created and the semantic and syntactic variables used in classifying tasks. Students were described by whether they grouped tasks by operations (addition, subtraction, multiplication, division, factoring), style of presentation ("textbook" exercises, one term to find, two terms to find, three terms to find), type of algebraic expression (polynomials, algebraic fractions, radicals, complex numbers), or some combination of the three. The information obtained from this test was also used in the analysis of the student's ability to transfer algebraic operations.

## Mathematical Memory Test

The Mathematical Memory Test (see Appendix C) was used to describe the students' memory for the features of algebraic tasks they had seen but not recently solved. The Mathematical Memory Test was based on instruments used by Silver (1978) and Kantowski (1977). The Mathematical Memory Test was developed through two pilot studies which are described in Appendix A. It used the same tasks as the Classification Test.

The Mathematical Memory Test was administered by the experimenter immediately after the student had completed the Classification Test. The Mathematical Memory Test consisted of two parts. In the first the student was asked to recall the features she used to classify tasks. In the second she was asked to recall as much as possible about the cards with the circled numbers in the right-hand corner. The student's responses to the Mathematical Memory Test were audio-recorded. The experimenter provided numerous hints in an effort to get as complete a report as possible.

The information obtained from this test was used to describe the students and in addressing the research question concerning perception of and memory for algebraic tasks. The semantic and syntactic attributes of the tasks that were recalled were of interest.

## Preference for Algebraic Problems Test

The Preference for Algebraic Problems Test (see Appendix C) was used to obtain information on the relationship between a student's classification of algebraic tasks and her preference for solving these tasks. The tasks were the same as those used in the Classification and Mathematical Memory Tests. The Preference for Algebraic Problems Test was based on an instrument used by Kantowski (1977) in her investigation of junior high-school students' problem-solving processes. The test was not pilot tested.

After the student had completed the Classification and Mathematical Memory Tests, the 21 cards were shuffled and returned to the student. She was requested to place the cards in order of preference: "Put the one you 'like' most first." When the Preference for Algebraic Problems Test was completed, the student was asked to go through the cards one-by-one and explain her ordering.

The descriptive information obtained from this test was used to provide evidence of a student's self-perception and attitude toward specific types of tasks (such as, for example, complex numbers). An examination of the relationship between a student's classification scheme for particular algebraic tasks and preference for these tasks was of value for describing the students and in addressing the research question concerning attitudes towards algebra.

### Information Sheet

The final instrument used to gather descriptive data was an information sheet (see Appendix C). The student was requested to provide background information concerning previous mathematics courses taken and the final grade received. Also requested were comments concerning goals, background, and expectations with respect to the basic algebra course. This information was used to help describe the subjects.

## Series of Algebra Interview Tasks

One of the limitations of Breslich's (1939) research discussed in Chapter II was his use of textbook exercises to gather information on the algebra student's thinking processes. The students' verbalization of their solutions limited the analysis of their thought processes in two regards. First, students often knew a procedure so well that they would jump over their reasoning during the thinking aloud process. This is similar to asking an adult to think aloud while solving "5 + 2." A second limitation of the use of textbook exercises was the tendency for students to state "rote" verbalizations of their thinking process. In these situations the process may have been rotely learned and, in fact, not be part of the students' thinking process.

Ausubel (1980) noted that in order to obtain evidence of meaningful learning "test questions must be phrased in a different language than that in which the material was first encountered" (p. 244). To study understanding, new types of tasks had to be created that were closely related to the algebraic content of the course and that were problems for the students.

In an effort to develop problems related to the exercises of algebra the concept of a "missing addends task" was adapted to algebraic material. The "missing addends" task is used in the primary grades as a transition from addition to subtraction. Because of the style of its presentation in many textbooks, it has come to be perceived as a problem by many children.

The tasks used in the interviews (see Appendix D) were selected from four topics of the basic algebra curriculum: operations with polynomials, operations with algebraic fractions, operations with radical expressions, and operations with complex numbers. Each topic identified a separate series of 16, 16, 11, and 8 tasks, respectively. The tasks from each series were of four types:

1. Tasks of the form □ * b = c or a * △ = c in which * was one of the basic operations; a, b, c were given algebraic expressions, and □ and △ were algebraic expressions the subject was requested to find. These tasks each had a unique solution.

2. Tasks of the form □ * △ = c in which * was one of the basic operations, c was an algebraic expression, and □ and △ were algebraic expressions the subject was requested to find. These tasks might or might not have unique solutions.

3. Tasks of the form □ * △ = ○ in which only the operation * and certain general conditions for the unknown algebraic expression □, △, and ○ were identified and the

subject was requested to find appropriate algebraic expressions to meet the given conditions. (For example, "Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$ whose quotient is a positive integer.") These tasks did not have unique solutions.

4. Tasks designed to ascertain whether the subjects knew and understood the vocabulary and symbolism used in the interviews sufficiently to solve the other tasks. If the subjects did not understand the vocabulary or symbolism used in the interviews, instruction was provided.

A classification of the tasks used in the study in terms of the problem type, topic, and operation is presented in Table 3.2. For example, Task 12 of the Algebraic Fractions Series involves finding two algebraic fractions whose difference is given in the task. The Polynomial Series provided baseline data on the student's ability to generalize to unknown instances, reversibility, and flexibility of approach. The Algebraic Fractions Series provided further evidence of reversibility, flexibility, and ability to apply generalizations in complex situations. The Radical Expressions and Complex Numbers Series provided evidence of the ability to transfer algebraic operations within the content of basic algebra.

The items in these series were tried out in the second pilot study (see Appendix A) individual interviews and on paper-and-pencil assignments for an entire basic algebra class. The pilot study showed that the tasks would be

Table 3.2

Classification of Algebraic Interview Tasks by Series,

Type of Problem, and Operation Involved

(Entries are task numbers within a series.)

TYPE OF PROBLEM

| | Find the Missing Algebraic Expressions: | | | | | | Vocabulary and Algebraic Symbolism |
|---|---|---|---|---|---|---|---|
| **SERIES** | | One Expression $\square \circ B = C$ or $A \circ \triangle = C$ | | Two Expressions $\square \circ \triangle = C$ | | Three Expressions $\square \circ \triangle = \bigcirc$ | |
| | - | 14 | - | 8 | - | 2 | |
| Polynomials | x | 12, 13 | x | 9, 10 | x | 4, 11 | 1, 3, 5, 6, 16 |
| | | | ÷ | 7 | ÷ | 15 | |
| | + | 1, 5, 11, 17 | | | | | |
| Algebraic Fractions | - | 4, 8, 15 | - | 12, 13 | | | |
| | x | 2, 6, 14, 16 | | | | | |
| | ÷ | 3, 7, 9, 10 | | | | | |
| | + | 4 | | | | | |
| Radical Expressions | - | 9 | | | | | 1, 2 |
| | x | 3, 5, 6, 7, 11 | | | | | |
| | ÷ | 8, 10 | | | | | |
| | + | 1 | + | 7 | | | |
| Complex Numbers | - | 4 | - | 5 | | | |
| | x | 2, 3 | | | x | 6 | |
| | | | | | ÷ | 8 | |

problems for most basic algebra students and that individual interviews would be rich in qualitative data pertinent to the research questions.

## How? The Procedures of Investigation

The present study was designed to provide information on basic algebra students' understanding of content that was previously covered in the course. The processes these students used in solving algebraic tasks were of interest. To catch the processes as they occurred a clinical interactive methodology, the ascertaining experiment, was used.

### Selection Procedures and Initial Sessions

The study was divided into three phases. In the first phase the participants were selected and initial descriptive data was obtained. The initial set of sessions was divided into two parts. First, small groups of students from the basic algebra course completed the information sheet and the CSMS Algebra Test. As they completed the test the students individually discussed these two descriptive instruments informally with the experimenter. The purpose of the discussion was to get a feel for each student's ability to verbalize thoughts as a criterion for inclusion in the study. These sessions were conducted during the third week of classes in the spring quarter, 1978.

In the second set of sessions, the Generalization Test was administered individually. After completing the test, the student was audio-recorded as she discussed the test with the experimenter. In addition to information concerning the subject's perceptions and attitudes with regard to the algebraic tasks, the discussion provided an opportunity for the subject to become accustomed to the idea of being audio-recorded while talking with the experimenter.

## Interviews

The interviews constituted the main, long-term portion of the study. Each subject participated in an average of two 45-minute interview sessions each week for the next five weeks. Each subject met individually with four interviewers: the experimenter, two doctoral students, and a faculty member in the Mathematics Education Department of the University of Georgia. All four interviewers were experienced instructors of basic algebra.

The interview sessions were held in a private office on the campus of the University of Georgia. A reel-to-reel recorder and a microphone were in prominent view of the subject. The interview procedure was developed in the second pilot study (see Appendix A). Minor adjustments were made on the basis of the pilot study. For example, reel-to-reel tapes were used in lieu of cassettes because of experience with faulty cassettes.

The tasks were given to the subject one at a time; with only one problem on a sheet and room for the subject to write. The subject was not expected to complete all of the items on each series of tasks. The order of presentation varied according to the speed at which the subject worked and the time available for a series. Before each session the tasks were ordered for the interviewer to present to a subject. If a subject did not complete all the tasks planned for a given session a subset was chosen for the following session. In order to assure that each subject solved a common core of problems, core tasks were put first for some students.

The subject was requested to "think aloud" as she attempted to solve each problem. If a subject would lapse in her verbalization, she was encouraged to tell what she was thinking. When a student appeared to be having a lot of difficulty, hints were provided. At first, the hints were general ("What are you trying to find?" "What's giving you a problem?"), but gradually they increased in specificity. In some instances, the hint consisted of pointing out a particular error, and in extreme cases the interviews were instructional. A problem was rarely left incorrect or incomplete. The long-term nature of the investigation allowed one to trace the effects of this interaction on the development of the student's processes in solving algebraic tasks.

The interviews were flexible. If an interviewer noticed something of interest in a student's response, the interviewer would create new questions to follow the direction of the student's thought. For example, one student's dependence on rules in answering Task 1-4, (Find two binomials whose product is a binomial), led to the follow-up problem of finding two binomials whose product had four terms.

Two kinds of questions frequently asked by the interviewer were:

1) Is your answer unique? How can you tell?

2) Could you have figured out the answer another way?

These questions focus, respectively, on the generalizability of responses and the flexibility of approach. Here, too, the subjects were occasionally prodded to demonstrate both types of reversibility (negation and compensation).

The area of content tested in the interview sessions trailed its study in the classroom by from 2 to 4 weeks. The Polynomial Series was used in four sessions. The Algebraic Fractions Series was used in two sessions. The Radical Expression Series and Complex Number Series were used in one session each. One additional session was used to permit students to complete tasks from series in which few problems had been completed. This generally involved the Algebraic Fractions Series. The problems in that series tended to take students much longer to solve.

### Testing and Descriptive Data

In the final phase the subjects completed the Classification, Mathematical Memory, and Preference for Algebraic Problems Tests. These tests were used to describe the subjects' behavior at the end of their course and after completing the interview tasks. Additional descriptive data were collected in the form of the regular classroom tests on the same content as the interviews.

## To Whom? The Selection of the Subjects

### The Pool of Subjects

Each quarter there are approximately 14 sections of basic algebra offered at the University of Georgia. About 30 students are enrolled in each section. The majority of the students are required to enroll in basic algebra because they failed the course exemption test given to all freshmen entering the University of Georgia. The test used in 1977-78 was created by a member of the Department of Mathematics and was designed to identify those students who require additional training before enrolling in what the department views as college level mathematics. A student is required to answer correctly at least 60 percent of the items on the exemption test in order to exempt basic algebra. As a form of security for the validity of the classification of students, the exemption test was not available for further description.

Basic algebra is designed to prepare students for college mathematics. Its topics include: basic operations with monomials, polynomials, algebraic fractions, and expressions involving radicals and complex numbers; solving linear and quadratic equations and inequalities; graphing functions; and solving systems of linear and quadratic equations and inequalities by elementary means. By design, this course is roughly equivalent to an upgraded first-year high school algebra course. The topics are the same but the examples given in basic algebra at the college level are more complex. Another major distinction is that at the University of Georgia about 50 sessions are allotted for presenting the material in a basic algebra course, while about 180 sessions are provided in a high school algebra course.

Of the more than 400 students who register for basic algebra each term, nearly all have had at least one year of algebra and one year of geometry in high school. Many students have also had a second and possibly a third year of high school algebra. The subjects used in this investigation were enrolled in basic algebra during the 1978 spring term. The textbook used in the course was by Gaughan (1970). This investigation of student's understanding of basic algebra was made with college students, rather than high school students, because of the practical advantage of scheduling the interviews on campus over a period of 2 months.

## How the Subjects Were Selected

During the second week of the 1978 spring term, a "call for assistance" (see Appendix B) was distributed to the students in five sections of basic algebra. These sections were chosen because their instructors were all experienced teachers of basic algebra and were either faculty or doctoral students in the Department of Mathematics Education at the University of Georgia.

Of the 116 students who returned the form, 27 indicated an interest in finding out more about the study. Eight of these students either did not show up for the initial interview or were unable to be contacted in time to participate in the study. Five students declined because of the length of time demanded by participation in the study.

At the initial interview the 14 students still interested in participation were told about the purpose and demands of the investigation. They completed a consent form (see Appendix B), an information sheet, and the CSMS algebra test (see Appendix C). After completing the instruments, the students were interviewed individually concerning their reactions to the instruments.

After the initial interview the student's instructor for basic algebra was contacted and asked to compare the student to other students the instructor had taught. Ratings were limited to strong, weak, or average. Eight subjects were chosen to participate in the study from four sections of basic algebra. Six of the students were identified as above average by their instructors. Two of the subjects were rated as below average by their instructors.

The subjects were selected on the basis of the recommendation of the instructor, the educational background of the students, and the ability of the students to discuss some of their solutions to items on the CSMS algebra test. Students who "froze" during the interviewing were not selected for participation. The students not included as subjects were sent letters of thanks (see Appendix B).

The students who participated in the study did so voluntarily. The only reward was that a portion of time at the end of each interview was set aside for tutoring. The schedule demanded 2 hours of interviews from each student each week. By the end of the second week of interviews two students withdrew from the study due to the length of time demanded. The present investigation was conducted with the four remaining students who had been identified by their basic algebra instructors as above average. Each of the four remaining students was female.

## A Description of the Students

The students were requested to allow information from their permanent records and their instructors to be used in the study (see Appendix B). The information from these sources is presented for each of the four subjects (Emily, Lynn, Claire, and Patty) in the following sections. In these sections, the subjects are further described by their responses to the background information sheet, the CSMS Algebra Test, the Generalization Test, and the course tests.

### Emily

Background information. At the time of the study Emily was a freshman at the University of Georgia, enrolled in the School of Arts and Sciences and specializing in Pre-Journalism (Advertising). During her first two terms at the University, she acquired a 3.25 cumulative grade point average. Her parents were both college graduates. Her father was a professor in the School of Veterinary Medicine at the University of Georgia, and her mother had received a B.S. degree in nursing.

Emily was enrolled in a basic algebra course because she had failed to pass the exemption test given to entering freshmen during the summer orientation. During her initial interview with the investigator, she indicated that her reason for volunteering for the present study was because she felt that it would be of value for the next generation.

Emily had three years of high school algebra: Algebra I (with a B received as her final grade), Algebra II (with a B received as her final grade), and Algebra III (with a C received as her final grade). In addition to these mathematics courses, she had a course in Geometry (with a C received as the final grade). Her high school grade point average was 3.40. On her high school quarterly reports, she had received only 5 C's and each of these was in mathematics. She scored a 540 on the verbal component and a 580 on the mathematics component of the SAT examination.

After 3 weeks of the spring quarter Emily was described by her basic algebra instructor as a strong student. At the completion of the course the instructor commented that Emily "did not need to be there." Her final grade in basic algebra was an A and her test scores relevant to the algebraic content of the study were 93, 100, and 100 percent. Copies of the three tests are included in Appendix E to provide the reader with a sense of the relationship between the content presented in the basic algebra course and the content of the interview tasks included in the study. Emily made four errors on the first test. She classified these as "dumb mistakes." The errors included some confusion with the term "numerical coefficient," subtracting only the first term of a binomial, forgetting to evaluate the second term of a complex expression, and not completely factoring an algebraic expression. In the latter case, she factored $y^3 - 9y^2 + 20y$ into $(y^2 - 4y)(y - 5)$. On the tests she demonstrated an ability to perform the basic operations with algebraic expressions (including tasks with variable exponents); to factor algebraic expressions (including a sum of two cubes); to perform the basic operations with algebraic fractions; to simplify radical expressions and perform the basic operations with radical expressions; and to simplify and multiply complex numbers. In general Emily solved each problem in a very orderly and precise fashion.

CSMS Algebra Test. Emily was able to interpret letters in all six variants presented in the CSMS Algebra Test. In general, she had little difficulty with the meaning of variables. She made errors on only 5 of the 51 responses requested on the test. In terms of Küchemann's (1978) classification scheme for the different ways letters are used in algebra, her errors were made in evaluating letters, in the use of letters as an object, in the use of letter as a specific unknown, and in the use of letter as a generalized number. These errors represented isolated mistakes.

Emily's errors indicate some confusion with the meaningful use of letters in algebra. For example on Item 6 (What can you say about b if b + 2 is equal to 2b?), she responded "2b - 2 = b". Rather than evaluate b, her response defined it in terms of itself. On Item 19i (a = b + 3. What happens to a if b is increased by 2?) and Item 19ii (f = 3g + 1. What happens to f if g is increased by 2?), she responded "a = b + 5" and "f = 3g + 7", respectively. The variables b and g were apparently of constant value for Emily, since she wrote the new value for a and f in terms of the original values for b and g. Although these errors were isolated, they provide some indication of potential areas of difficulty in Emily's understanding of algebra.

Generalization Test. On the Generalization Test (see Appendix C), Emily experienced difficulties created by the increased complexity of the variants of a given rule and the interference caused by related rules on the application of a given rule. She experienced no apparent difficulty in applying Rule 1, $(a^n)^m = a^{nm}$, in any of its variants. However, during the application of Rule 2, $(A + B)^2 = A^2 + 2AB + B^2$, some confusion did arise as the variants became more complicated:

4) $(a^3 + b)^2 = a^5 + 2a^3b^2 + b^2$

5) $(2x^4 + y)^2 = 4x^{\overset{6}{\cancel{8}}} + 4x^2y^2 + y^4$

6) $(5 + 3x^2)^2 = 25 + 30x^2 + 9x^4$

7) $(a^n + b)^2 = a^{\overset{2+n}{\cancel{2n}}} + 2a^nb + b^2$

8) $(x^{3n} + 1)^2 = x^{3n+2} + 2x^{3n} + 1$

As indicated in Items 5 and 7, Emily was internally debating between whether to add exponents or multiply them when squaring a term. Although the false generalization won out, her first instinct in Item 5 and again in Item 7 provide evidence that the debate continued. The Generalization Test was administered after she had already completed her test on operations with polynomials in her basic algebra course. This test is included in Appendix E. In discussing her completion of the Generalization Test, Emily described the difficulty she had with exponents as follows: "I was getting mixed up on exponents but I think that's because we're working on them now and my mind's -- all the rules are in there, but they're not straightened out yet." Perhaps these errors on the Generalization Test reflect a stage of new confusion created by the input of new rules. This same effect was noted by Brownell (1939) in his studies of children's understanding of arithmetic.

In addition to the interference created by newly learned rules on past rules, Emily reported an interference in the application of a new rule created by a previously learned rule. In attempting to apply Rule 3; $(a, b) + (c, d) = (ad + bc, bd)$, she noted some difficulty in its similarity with her method for multiplying binomials: "One thing on this 'ad'; you know you're used to doing the FOIL and it's 'ac' and then 'ad'. And I found myself wanting to start off with 'ac' there."

Emily applied Rule 3 successfully in all but the last variant presented on Page 3 of the Generalization Test:

$$(1/2\ x,\ x^n) + (x^n,\ 1/2\ x) = (1/4\ x + (x^n)^2,\ 1/2\ x^n).$$

Her error in multiplying $(1/2\ x)(1/2\ x)$ might be explained by her attention having been maintained on the more complex aspects of the task.

Emily experienced difficulty on Page 4 of the Generalization Test. On this page both Rule 3; $(a, b) + (c, d) = (ad + bc, bd)$ and Rule 4; $(a, b) \cdot (c, d) = (ac, bd)$; are provided. After correctly applying Rule 3 in two instances, she mixed the two rules in her solutions for the next three tasks:

3) $(x, 3) \cdot (2, x) = (x^2, 6)$

4) $(2, 3a^2) \cdot (3, 2a^3) = (4a^3, 9a^2)$

5) $(1/2\ x, x^n) \cdot (x^n, 1/2\ x) = \cancel{1/2 x^{2n}} \quad (1/2\ x^2, x^{2n})$.

When she switched back to the addition algorithm, Rule 3, which she had generally applied correctly on the previous page, she again mixed the operations:

6) $(3a, ac) + (2a, bc) = (3a + bc + 2a + ac, abc^2)$
$(5a + bc + ac, abc^2)$.

On Task 7, she apparently recognized her error and corrected it:

7) $(a, a + b) \cdot (b, a + b) = (ab, a^2 + 2ab + b^2)$.

This analysis is also supported by the following comments Emily made upon her completion of the Generalization Test:

> It wasn't difficult. I mean, you had to pay attention - like here [Page 4] where you mixed the signs. Cause I found myself going off and saying, 'Wait! That's a multiplication!" And one thing that I kept getting mixed up on is like here it's 'ac' and up here it's 'ad' and I probably made some mistakes. I caught myself starting to do it a couple of times.

In the midst of "catching herself" on Task 9 of Page 4 of the Generalization Test, Emily multiplied $x \cdot x^2$ and obtained $x^2$. This error might be explained by her concentration on the differences between the two rules.

## Lynn

Background information. At the time of the study Lynn was a sophomore at the University of Georgia enrolled in the College of Arts and Sciences with an unspecified major. Her cumulative grade point average for her first five quarters at the University was 2.67. Upon entering the University she was awarded advanced placement for two English courses, one History course, and four Spanish courses.

Lynn had two years of high school algebra: Algebra I (with an A received as her final grade) and Algebra II (with a B received as her final grade). These were the only mathematics courses in which she enrolled in high school. Her high school grade point average was 3.57. She received final grades of A or B in all of her high school subjects and was invited to participate in the Georgia Governor's Honors Program, specializing in "Band." She scored a 420 on the verbal component and a 440 on the mathematics component of the SAT examination.

Three weeks into the spring quarter, Lynn was described by her basic algebra instructor as a good student. At the completion of the course, the instructor commented that she "was very quiet in class, would give explanations way ahead of her classmates, but was not always conscientious about her homework and rarely came to review sessions." Her final grade in basic algebra was a "borderline" A and her test scores relevant to the algebraic content of the study were 94, 99, and 93 percent. Lynn made three errors on her first test. Her first two errors involved recalling and applying the definitions for "the degree of a polynomial" and for "numerical coefficient." Her third mistake was made in multiplying $2a^n(3a - 7b^3)$. Her answer was $6a^{n+1} - 14ab^{n+3}$. On this first test she demonstrated an instrumental understanding of the basic operations on polynomials, including tasks with variable exponents. She also showed that she knew how to factor polynomials, including a sum of two cubes.

On her second test in basic algebra, Lynn demonstrated an instrumental understanding of the basic operations on algebraic fractions. Her only error on the test was made on a task involving multiplying three algebraic fractions with monomial numerators and denominators. In copying her solution from her work, she neglected to copy the numerical coefficient of her denominator.

On her third test in basic algebra, Lynn demonstrated an instrumental understanding of the basic operations on radical expressions including how to simplify radical expressions and rationalize denominators. She also demonstrated that she was able to rewrite radical expressions with negative radicands as complex numbers in the "a + bi" form. Her only error on the test was on the task: $(2 + 3i)(5 - 4i)$. She wrote the following on her test paper:

$$\begin{array}{l} ac - bd)(ad + bc) \\ 10 - 12)(-8 + 15) \\ \quad\; -2 \qquad\quad 7i \end{array}$$

CSMS Algebra Test. Lynn was able to interpret letters in all six variants presented in the CSMS Algebra Test. She made errors on 6 of the 51 responses requested on the test. The errors involved situations in which a letter was used as a specific unknown, as a generalized number, and as a variable.

Of the six errors made by Lynn on the CSMS Algebra Test, three of them can be attributed to a difficulty she had in coordinating more than one condition in the solution of a problem. For example, she did not answer Item 14 (What can you say about r if $r = s + t$ and $r + s + t = 30$?) or Item 16 (What can you say about c if $c + d = 10$ and c is less than d?). On Item 19 ($f = 3g + 1$. What happens to f if g is increased by 2?), she was content with answering "f is increased". On Item 13 (Write $3a - b + a$ more simply.), she may have treated the problem as if parentheses were forgotten or as if the negative sign referred to both the latter terms and simplified the expression as $2a - b$. The error made on Item 3 (Which is larger, $2n$ or $n + 2$?) may have been caused by her having thought of numbers that satisfy her statement of the solution ("2n is larger because you're taking some number twice, whereas with n + 2, you're only adding 2 more to a number."). The final error made by Lynn on the test was on Item 18ii (When is $L + M + N = L + P + N$ true?). She responded, "Never." She apparently assumed that $M \neq P$. This was consistent with Wagner's (1977) findings mentioned in Chapter II.

Generalization Test. Lynn had mixed success on applying a rule to increasingly more complicated algebraic expressions. In applying Rule 2, $(A + B)^2 = A^2 + 2AB + B^2$, she developed a "bug" in her application of the rule. The following were selected from her "stair-case" of applications of Rule 2:

3) $(5x + y)^2 = 25x^2 + 10xy + y^2$
4) $(a^3 + b)^2 = a^9 + 2a^3b + b^2$
5) $(2x^4 + y)^2 = 4x^{16} + 4x^4y + y^2$
8) $(x^{3n} + 1)^2 = x^{6n} + 2x^{3n} + 1$

The "bug" occurred in cases where a variable factor of the term was raised to a numeric power greater than 1. For example, in squaring $a^3 + b$ and $2x^4 + y$ she treated $(a^3)^2$ and $(2x^4)^2$ as if they were equal to $a^{3^2}$ and $2^2x^{4^2}$. This false generalization of the rule $(a^m)^n = a^{m^n}$ was only applied in cases where the exponents are numeric and greater than 1.

Initially Lynn perceived Rule 3, $(a, b) + (c, d) = (ad + bc, bd)$ to be $(a, b) + (c, d) = ad + bc + bd$. It was not until her seventh application of the rule that she realized her mistake and altered her previous applications of the rule. Even in these applications she did not include the parentheses in her answers. With the exception of the deleted parentheses, she applied the rule correctly in all its variants, but after applying it she made two errors in attempting to simplify two of her algebraic expressions. On Item 5, $(1, a^3) + (3, a^2)$, she added $a^2 + 3a^3$ and obtained

$4a^5$ and on Item 6, $(2, x^n) + (6, x^n)$, she added $2x^n + 6x^n$ and obtained $8x^{2n}$. On Item 7, $(a, a + b) + (b, a + b)$, she "caught herself" multiplying ab instead of $a(a + b)$. The visual stimulus of the syntactical similarity of Rule 3 with the multiplication of binomials apparently set off an incorrect procedure which she noted and corrected.

The presentation of Rule 3, $(a, b) + (c, d) = (ad + bc, bd)$, and Rule 4, $(a, b) \cdot (c, d) = (ac, bd)$, together created only minor interference for Lynn's application of the rules. She began her solution to Item 6, $(3a, ac) + (2a, bc)$ by multiplying the first terms in the ordered pairs, but as soon as she wrote down $6a^2$ she realized her error and applied Rule 3 correctly. On Item 10, $(x^3, 3x^2)^2$, she began to apply Rule 3, but realized her mistake and applied Rule 4. In applying Rule 4 to Item 10 she erroneously wrote her solution as $x^6 + 9x^2$. She also erred in her application of Rule 3 to Item 8, $(3a + 2b, a) + (2a, bc)$. Rather than multiplying $(3a + 2b)bc$, she apparently just multiplied $(3a)(bc)$ and obtained "$3abc + 2a^2$, abc."

Although Lynn indicated she was currently studying algebraic fractions in her basic algebra course, she made no connection between Rule 3 and Rule 4 and the addition and multiplication of algebraic fractions. Instead, she at first compared Rule 3 and Rule 4 to the rules for the properties of algebra. This may account for her initial mistake of writing the sum of ad + bc + bd as her application of Rule 3. She compared Rule 4 to the multiplication of sum and difference of two terms, "where you don't get middle terms."

Claire

Background information. At the time of the study Claire was a freshman at the University of Georgia enrolled in the College of Arts and Sciences with an English major. Her cumulative grade point average for her first two quarters at the University was 3.83. She was on the Dean's List both quarters. Upon entering the University Claire was awarded advanced placement for two French courses and one History course.

On the information sheet, Claire described herself as follows:

> I am an English major with very weak mathematical abilities which I blame on a poor elementary school background. I don't hate math anymore, but I want to do well in the lower maths that I am required to take. I enjoyed geometry better than algebra in high school.

Claire had two years of high school algebra: Algebra I (with a B as her average grade) and Algebra II (with a C as her average grade). In addition to these mathematics courses, Claire had a year long course in Geometry (with an A as her average grade) and a quarter year course on statistics (with an A as her final grade). Her high school grade point average of 3.71 was sufficient to rank her in the top 10% of her graduating class. Throughout high school

Claire had received only four C's on her quarterly report, and each of these was in mathematics. Claire scored a 630 on the verbal component and a 410 on the mathematics component of the SAT examination.

After three weeks into the spring quarter, Claire was described by her basic algebra instructor as an "above average" student. At the completion of the course the instructor commented that Claire "worked very hard, put a lot of time in, participated in class with comments such as, ' I couldn't get this.' You could tell she tried it first. Claire always had her work with her and would come in out of class. She appeared to lack self-confidence." Claire's final grade in basic algebra was a B, and her test scores relevant to the algebraic content of the study were 75, 92, and 93 percent.

Claire made 12 errors on the first test. Her first error indicated an apparent misconception of the meaning of "numerical coefficient." Her next two mistakes were made by incorrectly copying an algebraic expression while evaluating the expression with given values for the variables. Two other errors resulted from having the signs incorrect in the solutions to two factoring problems. Two more mistakes were made while subtracting polynomials: $5 - 2x - 3$ was equated to $- 2x - 2$ and $[(x + 2y) - (x - 3y)] - [(x - 2y) - (x + 3y)]$ was rewritten as $x + 2y - x + 3y - x + 2y - x - 3y$. Another mistake, related to the latter one, was made in translating the task "Subtract $2x^2 + 3x - 1$ from the sum of

$2x^2 - 3x + 5$ and $x^2 + x + 2$." After correctly finding the sum to be $3x^2 - 2x + 7$, Claire neglected to place the trinomial $2x^2 + 3x - 1$ in parentheses and instead wrote $3x^2 - 2x + 7 - 2x^2 + 3x - 1$.

In addition to these "surface" errors, Claire made four errors that suggest the strength of several false generalizations in her understanding of algebra. The first was made while simplifying $2x^2 + 3x - x^2 + 5 + 3x - 5$. Claire apparently "canceled" the $x^2$ as she equated the above polynomial to $6x + \underline{2}$. Claire again "canceled" in a later problem:

$$\frac{x^n y^{n+2}}{x^n y^n} = 1^{2n+2}$$

On the problem $(x + y)^2 - (x^2 + y^2)$, Claire incorrectly squared the first binomial writing $x^2 + y^2 - x^2 + y^2$. Claire's final mistake on her first test in basic algebra was made in multiplying $2a^n(3a - 7b^3)$. Claire wrote the product as $6a^n - 14a^nb^3$.

At the same time she was making the above mistakes she was correctly performing the basic operations on polynomials. She correctly added, subtracted, multiplied, divided, and factored polynomials. For example, she flawlessly simplified $-[(u + 1) - 3(2u - 1) + 16]$ and correctly factored $y^3 - 9y^2 + 20y$.

On her second test in basic algebra, Claire demonstrated an instrumental understanding of the basic operations on algebraic fractions. Furthermore, she showed that she could perform the basic operations on the polynomials embedded in each algebraic fraction. She made four errors on the test. The first error was due to an apparent lack of understanding for the term "excluded values of the variable". Her second and third errors were made while reducing algebraic fractions to lowest terms. Claire wrote $\frac{-8xy^2}{24xy}$ in lowest terms as $\frac{y}{3}$ and later when subtracting $\frac{x}{x^2 - 36} - \frac{x + 1}{x^2 - 3x + 4}$, she did not reduce her answer $\frac{-x - 4}{(x - 4)(x + 4)(x + 1)}$. The last error Claire made on the test was on the following problem:

$$\frac{4}{k^2 - 25} - \frac{2}{k + 5} - \frac{k + 2}{5 - k}$$

Claire rewrote the denominator of the last fraction as follows:

$$\begin{aligned} (-1)5 &- K \\ -5 &+ K \\ K &- 5 \end{aligned}$$

No indication was made of where the (-1) came from. She made no other changes in the problem [illegible] compensate for its introduction. From this point on, she solved the problem correctly.

On her third test in basic algebra, Claire demonstrated an instrumental understanding of the basic operations or radical expressions including how to simplify radical expressions and rationalize denominators. She also demonstrated that she knew how to rewrite radical expressions with negative radicands as complex numbers in the "a + bi" form and how to multiply two complex numbers in the "a + bi" form. Her only error on the third test was made on content not included in this study.

CSMS Algebra Test. Claire had some difficulty with the meaning of variable as measured by the CSMS Algebra Test. She made 10 errors among the 51 responses on the test. The errors involved situations in which letters were ignored, used as variables, used as a specific unknown, and used as an object. The items on which she made errors and her solutions are included in Table 3.3.

Claire's concept of the meaning of variables as demonstrated in her errors on Items 2, 4, and 13 was inconsistent with her performance on her classroom test on polynomials. Unfortunately, the CSMS Algebra test was administered without observation and questioning. Therefore, it is impossible to clarify how these errors could be made by a student who had one week earlier scored a 76 points out of 100 on a test on polynomial operations.

Generalization Test. Claire compared the generalization test to the exercises in an algebra textbook:

> In algebra books, all the time, they give you just about the simplest one for the example. Then you get down into the exercise, and you've got this difficult problem, and you can't even relate the example to this problem because it's just so far removed.

Claire applied each rule with great care and precision -- matching each part of her solution to the statement of

## Table 3.3

### Claire's Errors on the CSMS Algebra Test

(Claire's solutions are in quotation marks.)

| ITEM | | |
|---|---|---|
| 2. | Write down the smallest and the largest of these:<br>$n + 1, n + 4, n - 3, n, n - 7$. | smallest "n"<br>largest "n - 7" |
| 3. | Which is the larger, $2n$ or $n + 2$? | "confused by question no response given" |
| 4. | 4 added to n can be written as $n + 4$. Add 4 onto each of these:<br>$n + 4$ "9" | n multiplied by 4 can be written as $4n$. Multiply each of these by 4:<br>$3n$ "12" |
| 13. | Write these more simply, where possible: | |
| | $2a + 5b =$ | "7ab" |
| | $3a - (b + a) =$ | "Not possible" |
| | $3a - b + a =$ | "a - b" |
| | $(a + b) + (a - b) =$ | "Not possible" |
| 21. | If this equation is true when $x = 6$ then what value of x will make this equation true? | $(x + 1)^3 + x = 349$<br>$(5x + 1)^3 + x = 349$ |
| | x= | "No relationship was made between the two equations" |

the rule and then simplifying the resulting expressions. For example, in applying Rule 1, $(a^m)^n = a^{mn}$, to Item 1, $(2^2)^2$, she first wrote her solution as $2^{2\cdot 2}$ and then equated her expression to $2^4$ or 16. On Item 6, $(a^{x+y})^2$, she apparently did not distribute the 2 as she equated the item to $a^{2x+y}$. Two other possible explanations are that she felt that her solution was the same thing as $a^{2(x+y)}$ or that she felt that in generalizing Rule 1 to binomial exponents only the first term of the binomial exponent would be affected by the power of the expression.

One "bug" developed in Claire's appplication of Rule 2, $(A + B)^2 = A^2 + 2AB + B^2$. In squaring a term, she squared the variable factors of the term, but not the numerical coefficient of the terms. For example, she solved Item 6 as follows: "$(5 + 3x^2)^2 = 5^2 + 2(5\cdot 3x^2) + 3x^{2+2} = 25 + 30x^2 + 3x^4$." As the example demonstrates, the "bug" did not involve constant terms.

Another "bug" was evident in Claire's application of the rules of the generalization test. On three separate occasions during her completion of the test, she equated $\frac{1}{2}x \cdot x^n$ with $\frac{1}{2}x^{2n}$. With the exception of these errors, she demonstrated an ability to apply generalizations in increasingly more complicated variants.

### Patty

Background information. At the time of the study Patty was a freshman at the University of Georgia, enrolled in the

College of Arts and Sciences and officially specializing in Music Theory. Unofficially she was in the process of changing her major, but was uncertain of what her new major would be. During her first two terms at the University, Patty acquired a 2.59 cumulative average. Her reasons for registering in a basic algebra course and feelings about mathematics are suggested in the following excerpt from her information sheet.

> The reason I'm taking this math course is because I didn't pass the test to exempt it! I am also hoping it will straighten out some of the confusions I've had in the past with algebra. Depending on how I do in this course, I'd like to get into some other math courses - hopefully Calculus.

The fact that Patty had not passed the exemption test was not consistent with the courses she had taken and the grades she received in high school. She had completed three years of high school algebra: Algebra I (in the junior high), Algebra II (with a C received as the final grade), and Algebra III/Trigonometry (with a B received as the final grade). In addition to these mathematics courses, She had taken courses in Geometry (B) and Computer Concepts (A). Her high school grade point average of 3.18 ranked her in the upper one-third of her graduating class (135/494).

Another incongruity with Patty's failure of the exemption test was her performance on the SAT examination. She scored a 640 on the verbal component and a 610 on the mathematics component. These scores were at the 99th and 95th percentiles, respectively, for a national sample of high school seniors. Patty's scores on the verbal and mathematics components of the SAT examination placed her in 95th percentile and 84th percentile, respectively, for students entering the freshman class of the University of Georgia in 1978. Unfortunately, neither Patty's SAT examination nor her University of Georgia exemption test was available for analysis.

Patty was described by her basic algebra instructor as a very good student. Her final grade for basic algebra was a B, and her test scores relevant to the algebraic content of the study were 97, 83, and 86 percent. Only one of the tests was made available for analysis. The test involved operations with radicals and some items from sections not included in the present investigation. Patty demonstrated on the test that she was able to simplify radical expressions involving square roots, rationalize denominators when necessary, and perform all four basic operations with radical expressions. In general Patty's solutions were very detailed. Most items involving radical expressions were solved twice as a check for validity.

CSMS Algebra Test. Patty was able to interpret letters in all six variants presented in the CSMS Algebra Test. She made errors on only 5 of the 51 responses requested on the test. One similarity across her errors was her focus on one condition of a problem in situations where more than one condition needed to be coordinated for a correct solution.

For example, on Item 14 (What can you say about r, if $r = s + t$ and $r + s + t = 30$?), she responded "$r = 30 - s + t$". Two points were of interest concerning her response. First, it was ambiguous. Since the CSMS Algebra Test was administered without observation, no questioning was used to determine whether parentheses were implicit in the response. Second, Patty may have focused on the second condition presented in the task and ignored the first condition. On Item 16 (What can you say about c if $c + d = 10$ and c is less than d?), she responded, "$c = 10 - d$." On this item she may have ignored the latter condition that $c < d$. Other possible explanations for these errors were that she viewed any "correct" response as appropriate and hence didn't search for a "best" solution or that she viewed the additional conditions as extraneous. For example, on Item 19 (Given $f = 3g + 1$, what happens to f if g is increased by 2?), she responded, "It increases."

Generalization Test. On the Generalization Test (see Appendix C), Patty easily generalized a given rule to the most complicated variants tested. She simplified resulting algebraic expressions except for expressions with variable exponents. The only "bug" noted in terms of her solutions involved situations in which the first term in a series was not properly squared. For example, she applied Rule 2, $(A + B)^2 = A^2 + 2AB + B^2$, as follows:

3) $(5x + y)^2 = 5x^2 + 2(5x)(y) + y^2 = 5x^2 + 10xy + y^2$

5) $(2x^4 + y)^2 = (2x^4)^2 + 2(2x^4y) + y^2 = 2x^8 + 4x^4y + y^2$

6) $(5 + 3x^2)^2 = (5)^2 + 2(5)(3x^2) + (3x^2)^2 = 25 + 30x + 9x^4$.

Patty did not square the coefficient of the first terms in Items 3 and 5 (although in the latter item she enclosed the first term in parentheses), but she did square the coefficient of the last term in Item 6. An error of the same type occurred in her application of Rule 3. In every item Patty correctly multiplied variables after their first occurrence.

### In Summary

The descriptive information presented here has been summarized in Table 3.4. The following characteristics were common for all four students. Each student received at least a B in her basic algebra course, with at least a B average on the tests relevant to the present study. Each student had at least two courses in high school algebra and received above average grades in other areas of the curriculum. Each student demonstrated (based on her test averages) a successful performance in her basic algebra course.

## Limitations of the Methodology

Hatfield (1979) cautioned that the qualities of a clinical study that provide its strengths are also the sources of its limitations.

Table 3.4

Description of the Subjects

| | EMILY | LYNN | CLAIRE | PATTY |
|---|---|---|---|---|
| Final Grade in Basic Algebra | A | A | B | B |
| Grades on Course Tests | 93 / 100 / 100 | 94 / 99 / 93 | 76 / 92 / 93 | 97 / 83 / 86 |
| Classification by Instructor | Didn't need to be there | Good -- not always conscientious | Above average -- a hard worker | Very good |
| University School and Major | Arts & Sciences<br>Pre-Journalism | Arts & Sciences<br>Undecided | Arts & Sciences<br>English | Arts & Sciences<br>Music Therapy |
| Classification | Freshman | Sophomore | Freshman | Freshman |
| UGA Cumulative Average | 3.25 | 2.67 | 3.83[a] | 2.59 |
| SAT Math Score | 580 | 440 | 410 | 610[b] |
| SAT Verbal Score | 540 | 420 | 630 | 640 |
| High School Mathematics Courses: Final Grades | Algebra I: B<br>Geometry: C<br>Algebra II: B<br>Algebra III: C | Algebra I: A<br>Algebra II: B | Algebra I: B<br>Geometry: A<br>Algebra II: C[c]<br>Statistics: A | Algebra II: C<br>Geometry: B<br>Algebra III/ Trigonometry: B<br>Computer Concepts: A |
| CSMS Algebra Test | 46/51 | 45/51 | 41/51 | 46/51 |
| Generalization Test - Rules Generalized | With Difficulty | With Mixed Success | Minimal Difficulty | Easily |
| Generalization Test - Rule Interference | Much Interference | Minor Interference | No Interference | No Interference |
| Other Remarks | Only C's in High School were in Mathematics | All A's and B's in High School<br>Governor's Honors (Band) | [a] On Dean's List Each Quarter<br>[c] Only C in High School | [b] 95 %ile in National High School Sample |
| | EMILY | LYNN | CLAIRE | PATTY |

> The flexibility which the interviewer enjoys in balancing a structured interview plan with spontaneous sidetracks initiated by either the adult or child allows for insightful exploration of the child's thinking. But widely disparate interviews create noise and confounding in the comparative study of individuals....Since clinical approaches are often time-consuming, researchers resort to fewer subjects. The advantage of longitudinal, detailed studies of fewer subjects is weighted against the possible loss of generality of any results. (pp. 25-26)

The following limit the generalizability and applicability of the study:

1. Subjects. The number of subjects is small (4) and they were selected in a non-random fashion. All four were female. All subjects were algebra students at one university and all used the same textbook in the algebra course. Although there were two different instructors, the course syllabus and common textbook reduced the variability of their experience in class.

2. Procedures. The interview procedure used to collect evidence allows considerable latitude for interpretation of the subjects' problem-solving processes. There was interference in their problem-solving processes in the form of hints and, at times, direct instruction by the interviewer. Requiring the students to "think aloud" may also have interfered with the subjects' problem-solving processes.

3. Tasks. The tasks, although closely related to the content of basic algebra were unusual in design. Students may have been intimidated by such verbal tasks, which may

have created a false impression of their ability to do more symbolic algebraic problems.

Because of these limitations, generalization of the results to other algebra students or other institutions is possible only to the extent that the students and the algebra courses may be similar. The students are described in detail in the preceding section. The algebra course in which they were enrolled may be judged by the presentation given in the textbook and the objectives emphasized on the instructor-made tests. Finally, the interpretations made of thought processes traced in the student's verbal solutions can be scrutinized by the reader in the protocols presented.

The generalizability of the results of this study depends on the degree to which the methodology and subjects have been adequately described to promote replication. Perhaps the most important "n" is the number of times a study has been replicated by other investigators under other conditions.

## Treatment of the Data

One characteristic of the ascertaining experiment is the experimenter's reliance on a qualitative, rather than quantitative, analysis of the data for determining and reporting results. In the present study the verbal protocols and written work of the subjects were scrutinized by the experimenter in terms of the questions of interest. Initially, the verbal and written responses of all subjects for each instrument were analyzed for regularities in observable sequences of behavior across subjects. Next, the responses for each individual were carefully examined with reference to the questions of interest for the present investigation. The results from these individual analyses are reported as case studies in Chapter IV. These results were supported by excerpts from the protocols and corresponding sections of the written work. Finally, the verbal and written responses of all subjects for each instrument were again analyzed for commonalities in observed patterns of behavior. These findings are reported in Chapter V.

The following list contains the subquestions used to focus the analysis of a student's written and oral protocols. The subquestions have been drawn from the questions offered in the discussion of the theoretical background and the review of the literature.

Question 1. What evidence of a reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

Question 2. What evidence of a flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

a) Is there a pattern to the student's behavior in terms of the types of reversibility? For example, are some operations, content, or problem types more likely to be approached by compensation than by negation?

b) If a student has just solved a problem correctly by one method, will she use the same approach to attempt to solve the next problem even when another approach is more efficient in terms of ease and accuracy? How flexible is a student in the use of negation and compensation?

c) If a student solves a problem with one approach, can she solve it another way? If the student uses negation to solve a problem, can she use compensation as well, and vice versa?

d) How persistent is a student in attempting to use a particular approach when experiencing difficulty in solving a particular problem? How does the student react to hints that suggest a different approach?

e) How does the student reverse her flow of thought on multi-step algorithms such as subtraction of algebraic fractions?

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

Question 5. What evidence of the use of false generalizations and false transfers is provided by someone who is successful in a basic algebra course?

a) To what degree of variation does the student extend her solutions? On tasks with multiple correct responses, does she make general statements explaining the solutions for a particular problem (deduce the general from the specific)?

b) Does the student generalize process or operations from one task to the next? Does she transfer the process or operations from a task in one content domain to tasks in another domain? Does she transfer an ability to factor polynomials to the factoring of radical expressions and complex numbers?

c) How does increased difficulty effect a student's application of generalizations? If she focuses on a particular difficulty, are incorrect connections more likely to occur? How stable (firmly established) are false generalizations? Is there a difference in her use of false generalizations during particular types of tasks?

Question 6. What evidence of the role of perception and memory on the solution of algebraic problems is provided by someone who is successful in a basic algebra course?

Question 7. What evidence does someone who is successful in a basic algebra course provide of her attitude towards algebra and her perception of her understanding of algebra?

a) How does a student's classification scheme for algebraic tasks relate to her preference for solving the tasks? What aspects of algebraic tasks does she tend to remember?

b) How confident is the student of her algebraic ability? What is her attitude toward the topics of algebra?

# CHAPTER IV

# FOUR CASE STUDIES

The individual case studies for Emily, Lynn, Claire, and Patty are presented in this chapter. The seven questions proposed in the first chapter are answered on the basis a qualitative analysis of each subject's written and oral protocols on the Algebraic Interview Tasks, CSMS Algebra Test, Generalization Test, Classification Test, Mathematical Memory Test, and Preference for Mathematical Problems Test. Transcribed protocols of exemplary sections of the student's solutions to the Algebraic Interview Tasks are presented in support of the analysis. Regularities of observed behavior among subjects are presented in Chapter V.

## Emily

### Reversibility and Flexibility

*Question 1*. What evidence of reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

*Question 2*. What evidence of flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

A Pattern of Behavior

Emily's approach to solving the algebraic tasks was characterized by her emphasis on negation. Of the 38 problems she attempted to solve by using a form of reversibility, only six were attempted by compensation (see Table 4.1). Four of these six problems involved multiplication of polynomials, and one involved multiplication of complex numbers. The application of compensation in these tasks involved factoring. One explanation for her switch from negation to compensation on these tasks was provided in her solution to Task 1-9 (Find two binomials, M(X,Y) and N(X,Y), such that $M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2$). She began her solution by choosing an arbitrary M(X,Y), X + Y and dividing it into $6X^2 - XY$: "I could divide it [X + Y] into that [$6X^2 - XY - Y^2$] to get the other one. Except I hate dividing. I think I've gotten in over my head. I know how to divide, I just can't get my mind on it." She switched her approach to the task: "I'm going to factor that [$6X^2 - XY - Y^2$]. I don't know why but I'm going to see what happens." She indicated some surprise that factoring worked.

The only instance in which Emily applied compensation to solve a problem that did not involve multiplication was on Task 2-12 (Create a subtraction problem involving two algebraic fractions in lowest terms with the same denominator and whose difference is $\frac{2A}{B}$). On this task, she

## Table 4.1

### Algebraic Interview Tasks Solved by Emily With Hints (Underlined) or Without Hints (Not Underlined) by Negation (N), Compensation (C), or Another (A) Approach

(The number in parentheses indicates the order of presentation for the task.)

| | Find the Missing Algebraic Expression: | | | Vocabulary and Algebraic Symbolism |
|---|---|---|---|---|
| | One Expression<br>□ * B = C or A * △ = C | Two Expressions<br>□ * △ = C | Three Expressions<br>□ * △ = ○ | |
| Polynomials | - 14(14,N)<br>x $\underline{12}$(12,N$^{*}$); $\underline{13}$(13,C$^{**}$) | - 8(8,N)<br>x 9(9,C); 10(10,C)<br>÷ 7(7,N) | - $\underline{2}$(2,A)<br>x 4(4,C); 11(11,A)<br>÷ 15(15,N) | 1(1,A); $\underline{3}$(3,A)<br>5(5,A); 6(6,A)<br>16(16,A) |
| Algebraic Fractions | + $\underline{1}$(20,N); $\underline{5}$(18,N)<br>- 4(17,N); 15(22,N)<br>x $\underline{2}$(21,N); 14(45,N); $\underline{16}$(19,N)<br>÷ 10(25,N) | - 12(23,C); 13(24,N) | | |
| Radical Expressions | + 4(29,N)<br>- 9(34,N)<br>x 3(28,N); $\underline{5}$(30,N); 6(31,N); $\underline{7}$(33,N); $\underline{11}$(43,N)<br>÷ 8(32,N); 10(42,N) | | | 1(26,A)<br>2(27,A) |
| Complex Numbers | + 1(35,N)<br>- 4(37,N)<br>x 2(36,N); 3(44,N) | + 7(40,N)<br>- 5(38,N) | x 6(39,C)<br>÷ $\underline{8}$(41,N) | |

* First attempted compensation

** First attempted negation

first set up the equation "$x - y = \frac{2A}{B}$" and then used the conditions of the problem to determine the numerators.

Emily generally attempted to solve each of the four sets of algebraic tasks by negation. On tasks in which she was requested to find one algebraic expression given an algebraic operation and two algebraic expressions, she would begin her solution by using inverse operations to rearrange the original problem. For example, as she finished reading aloud Task 1-14 (What trinomial subtracted from $5X^2Y - XY^2 + 7$ yields the difference $-X^2Y + 8$?) she planned her solution: "If I subtract this from that I should get the trinomial, 'cause this is the sum of those two, right. So I'm going to subtract here."

Emily immediately perceived of Task 3-3 (What number multiplied by $4\sqrt{2}$ equals $24\sqrt{6}$?) as an implied division: "So what you're really saying is $24\sqrt{6}$ divided by $4\sqrt{2}$ equals what [writes $\frac{24\sqrt{6}}{4\sqrt{2}} = X$]."

One explanation for Emily's application of negation to the algebraic interview tasks was that she viewed the tasks as equations to solve. Although it was difficult to determine the role of the variable in her solution to Task 3-3, her solution to Task 2-10 (Find the algebraic fraction in lowest terms if $\frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21}$ divided by it yields $\frac{2Y - 3}{Y - 7}$) provided further evidence of the methods in which she applied negation. She began the problem by translating the words of the problem into an algebraic equation: "Let's say, 'Find X so that if you divide it . . . so it says

$\frac{1}{X} \cdot \frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21} = \frac{2Y - 3}{Y - 7}$." Emily proceeded to multiply both sides of her equation by X. Then she multiplied both sides of the equation by $\frac{Y - 7}{2Y - 3}$ and simplified to solve for X.

Emily also used negation to solve many of the algebraic interview tasks of the form $\square * \triangle = C$ where $\square, \triangle$ are two algebraic expressions to be found, * is a given operation, and C is a given algebraic expression.

On Task 1-7 (Find two monomials whose quotient is $-5AB^2C$), she began: "Since that's the number I get when I divide them I could just multiply that by something and I'll get a bigger number that when I divide it by that that I multiplied it by I'll get that, which will be the quotient."

Emily adapted this procedure in her solution to Task 1-8 (Find two trinomials, P(X,Y) and Q(X,Y), such that $P(X,Y) - Q(X,Y) = -2X^2Y$): "I could, I think, work it backwards from the answer by -- no, I can't do that either -- I could add P, no wait. Okay I could make what Q is add it to that and that would give me P. I can do that." She initially chose $X^2 + 2X^2Y + Y^2$ as Q(X,Y) but quickly adjusted her selection when she realized that P(X,Y) would not be a trinomial.

Emily's solution to Task 2-13 (Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$) provided further evidence of her use of negation to solve tasks with two unknown algebraic expressions. She began, "This is going to be a little harder. No, it's not. Oh, if $x - y = \frac{2A}{B}$ then $y + \frac{2A}{B} = x$. So I could just make up a "y" and when I add it to that I'll get an x. That'll work . . . should work." She chose $\frac{6}{C}$ as her value for y and solved the task easily.

## Flexibility of Approach

For most tasks Emily viewed two forms of negation as the only appropriate ways to solve the tasks. After solving Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$), she was asked if she could think of another way to solve the task. She replied, "No, I mean you have to subtract. I mean there might be another way, but I don't know . . . I don't think . . . ." Repeatedly, she responded that the only possible method for solution was the immediate application of negation or setting up an equation. She perceived these to be "really basically the same thing" (Task 3-9). In cases where she applied negation without setting up an equation, she felt that she had just "skipped a step" (Task 4-3).

After Emily solved Task 4-7 (Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$ such that the sum of the two complex numbers is i), the interviewer tried to get Emily to solve the task by compensation. She had just used compensation to solve Task 4-6 (Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$ such that their product is a negative integer). The interviewer asked

Emily if she could solve Task 4-7 similar to the way in which she solved Task 4-6. She replied, "Well I knew a definite number that I had to get so I couldn't just make up two numbers by trial and error 'cause it would have taken me forever. So I guess I could have, but it'd be better to do it this way."

The following dialogue showed the difficulty Emily had in following the interviewer's hints when the interviewer tried to lead her through a compensation approach to Task 2-14 (Find the algebraic fraction in lowest terms if $\frac{X^2 - 4}{X^2 + X - 2}$ multiplied by it yields $\frac{X + 2}{X - 1}$):

> *Interviewer*. Is there any other way of working that problem?
>
> *Emily*. You know, you could set it up different saying this [$\frac{X + 2}{X - 1}$] . . . no this . . . this here [$\frac{X^2 - 4}{X^2 + X - 2}$] times some number equals that [ ], but then you'd end up dividing which is inverting and multiplying those two to get that.
>
> *Interviewer*. Just for the heck of it, write down what you were saying.
>
> *Emily*. You could set it up so you'd have $\frac{X^2 - 4}{X^2 + X - 2}$ times some number - I'll use A 'cause if you used X it would get confusing - equals $\frac{X + 2}{X - 1}$ and then you'd end up dividing both sides by this which is inverting and multiplying . . .
>
> *Interviewer*. Can you think of anything you might do without inverting and mutiplying?
>
> *Emily*. Ummm . . . No, 'cause you have to divide. You have to get all your numbers on one side and have your variable over here. And the only way to get rid of something that you're multiplying times something is to divide by it. I could . . . make a big fraction - put a big line there and put it down there - but I'd end up doing that anyway to solve it.

> *Interviewer*. I'm just wondering, is there any way you could change the form of that fraction . . . so that you might be able to get the solution without actually dividing?
>
> *Emily*. Gosh! Could I go ahead and factor it while it's over here and go ahead and cross things out?
>
> *Interviewer*. Try it.
>
> *Emily*. Oh no! . . . [writes $\frac{(X + 2)(X - 2)}{(X + 2 (X - 1)} A = \frac{X + 2}{X - 1}$ ] . . . But I don't know the rules for crossing out over an equal sign. Well I can cross those out [crosses out the x + 2's in the first fraction . . . I still have to divide. I wouldn't divide as big a number, but I'd still have to invert this and multiply it over here, wouldn't I?
>
> *Interviewer*. Well I was just wondering if there's any way you could look at this and think that you're going to multiply it times something and you want the answer to come out like that . . . [after a long pause] . . . You know that A has to be a fraction; so, for example, what could you multiply that fraction that you have there by, in fractional form, and have it turn out to be $\frac{X + 2}{X - 1}$.
>
> *Emily*. Okay, it looks like the bottom number would be one's, 'cause those two stay the same. But if it's one then it's really not a fraction. And x - 2 to get x + 2 . . . . If you multiply by -1 to change the sign it's going to change the sign on the front too. I can't see it. Not right off.

With a great deal of interviewer guidance and frustration, She eventually solved the task by compensation: "That's the same thing I got there -- How 'bout that!"

### Reversibility of Multiplication and Factoring

Emily's use of compensation was directly related to her recognition of the relationship between factoring and multiplying. For instance, in her solutions to Task 1-11 (Write a binomial P(X,Y) and a binomial Q(X,Y). What is the product of P(X,Y) and Q(X,Y)? Next, take the product you've just found and factor it), she first stated, "It would factor right back out to P and Q" and then picked specific binomials.

Factoring did not always work successfully for Emily. On her first chapter test in her basic algebra course, she demonstrated that at the time of the test she knew how to factor $X^3 + 8Y^3$. However, her solution to Task 1-12 (What trinomial multiplied by 4A + Y yields the product $64A^3 + Y^3$?) demonstrated the temporal nature of her application of the rule for factoring a sum of two cubes:

> What you could do is factor out a 4A + Y and find out what you have left . . . 4 goes into 64 - I don't know how many times . . . 4A + Y and that leaves 16A - that's not going to be a trinomial. Can I do that? . . . I'm trying to factor this and I'm just thinking if I'm doing it right. 64A . . . it say's trinomial . . . I can't do that because if I factor out a 4A there has to be a 4A in both terms - Right? - but if you multiply that by something it's got to be one of the factors. So there's got to be some way that I could factor where I guess the middle term cancels out. The only thing I know when a middle term cancels out is where you have 4A + Y and 4A - Y, but 4 times 4 is 16 not 64 and besides that's cubed. I'm thinking that there is a rule but I'm not remembering the rule so it's not doing me any good.

At this point the interviewer gave Emily a related question using natural numbers. She quickly solved the related question and Task 1-12 by negation.

## Reversibility of Multistep Algorithms

Emily applied negation smoothly to tasks that involved intermediary steps, such as operations with algebraic fractions. Her reliance on negation extended to tasks involving multistep algorithms that could be more easily solved by compensation, such as Task 2-1 (Find the algebraic fraction in lowest terms which when added to the sum of $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ yields $\frac{3X + Y}{Y}$). She attempted to solve the task by adding $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ and then subtracting the sum from $\frac{3X + Y}{Y}$. With the help of the interviewer, she was able to catch an error in her work and arrive at the solution. After Emily solved the task the interviewer asked her to look at the problem, but she never indicated any awareness of the relationship between her answer and the second addend.

Further evidence of Emily's use of negation in multistep tasks for which compensation would be easier was provided in her solution to Task 2-2 (Find the algebraic fraction in lowest terms which when multiplied by the sum of $\frac{3X}{5}$ and $\frac{X - 5}{15}$ yields 1). She apparently recognized the role of the multiplicative inverse in this task only after she completed the division: "Okay, first I have to add these two and then divide these two into 1 and I should get the third fraction." She indicated the sum of the first two fractions as $\frac{9X + X + 5}{15}$ and continued, "Then I would invert this and multiply it by 1, but I get that [$\frac{9X + X + 15}{15}$] when I do that. No, I get the opposite of this when I do that. Okay, when you divide . . . Okay, I want to divide one by this number so I would invert and mulitply it by one, which would give me the reciprocal of this. So the other fraction must be the reciprocal of this. It must be $\frac{15}{9X + X - 15}$. Right? That should be the answer." The interviewer questioned, "How do you know if your answers right?" Emily replied, "If I multiply these two together I should get one, which I would get. That makes sense, 'cause if you multiply . . . Yeah!"

## Ability to Generalize and Transfer

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

Question 5. What evidence of the use of false generalizations and false transfers is provided by someone who is successful in a basic algebra course?

### Ability to Deduce the General from the Specific

On tasks Emily solved by negation she generalized her solution to other examples by choosing a different algebraic expression for one of the unknown expressions. For example, she solved Task 4-5 (Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$ such that their difference is -2i) by setting up the equation $x - y = -2i$, solving for x, choosing a y, and determining what x was. When asked if her answer was unique, she replied, "No, because I made up the y. That could be anything and the x depends on what I make that or vice versa."

After solving Task 1-7 (Find two monomials whose quotient is $-5AB^2C$), Emily was asked, "If you were going to write the next problem sheet what kind of question might you ask?" She replied, "You're probably going to say, 'Find two binomials' or something like that and give me a quotient. But I'd do it the same way now that I've figured that all I have to do is mulitply that and see what I get and then that would be the two things."

On the multiplication tasks Emily solved by compensation, she was a bit unsure of how to generate other solutions. For instance, she obtained $A^3(A^m + 1)$ as her solution to Task 1-10 (Factor the following binomial into a product of a monomial and a binomial: $A^{m+3} + A^3$). In response to the question, "Could you find another answer?," she replied:

> I don't believe so. Again I'm not saying there's not a way, I just don't know what it is. I could switch the order but it's not another answer. No, I don't know of another way. Am I supposed to know of another way? Is it obvious? Am I supposed to see it? Let me look. I'm just seeing if there's anything else that I can factor out and I don't know 'cause I can't factor out an $A^m$ 'cause there's not an $A^m$ in both terms. But there's an $A^3$ in both terms, so that's the only thing that I see that I can factor. Oh, wait a minute. I could factor out like say $A^2$ and have $A^{m+1}$. No, yeah, . . . plus A. That's still a monomial and a binomial, but that's not factored completely 'cause you could still factor out an A.

Emily was asked if she could find another solution and she replied, "Just factor out an A and still have $A^{m+2} + A^2$." She indicated no other possible solutions.

On Task 2-12 (Create a subtraction problem involving two algebraic fractions in lowest terms with the same denominator and whose difference is $\frac{2A}{B}$) Emily noted that there were many possible solutions. When asked if all the solutions would have to have B as the denominator, she replied: "They'd have to be in lowest terms. Wait . . . I don't guess so because they could be in lowest terms and have the same denominator that could be different than B but then when you subtract them you might have something that you could reduce and get rid of something in the denominator and just leave B. So not necessarily would the denominator have to be B.'

### Ability to Generalize from Task to Task

Emily's solution to Task 1-13 (What trinomial multiplied by $A - B - C$ yields the product $A^2 - B^2 - C^2 - 2BC$?) provided some evidence of her ability to extend her generalizations of the processes and operations on polynomials. Initially she tried to solve the task by generalizing a process she applied to a previous task and dividing $A - B - C$ into $A^2 - B^2 - C^2 - 2BC$: "I need to do kind a like what I did before only this time I'll have three terms rather than two. I guess I can do it. I can try."

Emily had difficulty performing the long division. Despite the interviewer's efforts to guide her through the division process, Emily switched her approach to the task. She commented, "There's got to be an easier way." and began to try likely trinomials. Her choice of polynomials was apparently based on her attempt to generalize structure from one task to another. Her first choice was A - B - C and her second was A + B + C. She explained her choice of A + B + C as follows:

> Well I saw the squares there and so I said maybe it's kind a like the binomials where you have X + Y and X - Y and you get $X^2 - Y^2$. But it was just trial and error. I mean there wasn't really any great thought behind it.

Ability to Transfer

In Emily's solution to Task 4-2 (Find the complex number which multiplied by 3 - i equals 10), she demonstrated an ability to transfer processes and operations from polynomials, radical expressions, and equations to her work with complex numbers. She began her solution by setting up the equation x(3 - i) = 10. Next she divided both sides of the equation by 3 - i and then commented: "That's the same thing as having a radical in the denominator and you can't have that so you'd have to get rid of it. That's going to be 3 + i -- this is just a guess. I don't really know what I'm doing." She multiplied 3 + i in the numerator and denominator and wrote $\frac{30 + 10i}{9 + 1} = \frac{30 + 10i}{10} = 3 + i$. She continued, "So it equals 3 + i . . . But 3 + i times 3 - i doesn't equal 10! Yeah it does 'cause I just did it." She had not realized that she had already solved Task 4-2 when she rationalized the denominator, but continued simplifying $\frac{30 + 10i}{10}$. Even after simplifying she was a bit surprised at her solution. After completing the task she was asked if her solution was unique. She responded, "Ummm . . . yeah I think so. Real convincing there . . . You have to have $3 + i$ 'cause you have to cancel out the middle term and that's the only thing that'll do it." She had transferred the process of multiplying a sum and difference of two terms to multiplying two complex numbers.

Emily experienced a feeling of "lack of closure" in her solution to Task 3-4 (Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$). In an effort to achieve closure she attempted to transfer a variety of processes and operations from her experience with algebra. These attempted transfers included $\sqrt{3} + \sqrt{3} = \sqrt{6}$, trying to compare its solution to finding common denominators, factoring, multiplying one side of the equation $x + \sqrt{3} = \sqrt{6}$ by $\sqrt{3}$ and the other by $\sqrt{6}$. Although she suggested these possibilities, she quickly discounted each one. The false generalizations and false transfers she made in her solutions to the Algebra Interview Tasks were included in the next section.

## Errors, False Generalizations, and False Transfers

Most of the errors made by Emily on the Algebra Interview Tasks were self-detected and corrected. Besides computational errors, such as writing the product of Y + 2 and Y + 1 as $Y^2 + 3Y + 3$ (Task 2-16), her main error was in the translation of the conditions of a problem into an algebraic expression to simplify. On Task 1-2 (Write a binomial in X and Y where one of the terms is a constant. Subtract from your first binomial another binomial in X and Y where one of the terms is a constant. Simplify your expression by combining similar terms) she selected XY + 5 and 2XY + 3 as her binomials. Then she indicated her subtraction as XY + 5 - 2XY + 3 (without parentheses around the 2X + 3) and obtained -XY + 8 for her solution.

Emily made a similar error on Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$). She attempted to solve the task by negation. First she wrote $\frac{X}{X - 5}$ as the equivalent fraction $\frac{X^2 - X}{(X-1)\ (X-5)}$. Then she subtracted the numerators of the fractions, $2X - 15 - X^2 - X$ (without parentheses around the $X^2 - X$) and obtained $-X^2 + X - 15$ for the numerator of her answer.

Emily made another translation error in her solution to Task 2-16 (Find the algebraic fraction in lowest terms which multiplied by $\frac{Y^2 - 7Y + 12}{Y^2 + Y - 2}$ yields $\frac{Y^2 - 3Y + 4}{Y - 1}$). She began, "Oh I hate this! [Emily later indicated that she 'hated' all division problems, especially fractions] . . . I could divide. When you divide you invert and multiply. That makes sense. That's what I'm going to do." She inverted the $\frac{Y^2 - 3Y + 4}{Y - 1}$ and multiplied it by $\frac{Y^2 - 7Y + 12}{Y^2 + Y - 2}$ and obtained $\frac{Y - 3}{Y^2 + 3Y + 2}$ as her answer. In checking her answer she wrote $\frac{(Y - 3)(Y - 4)}{(Y - 1)(Y + 2)} \cdot \frac{(Y - 3)}{(Y + 2)(Y + 1)} = \frac{Y^2 - 7Y + 12}{Y^2 + Y - 2} \cdot \frac{Y - 3}{Y^2 + 3Y + 2}$ and then began multiplying the fractions: "I don't think this is going to work. I think I'm off somewhere. This isn't working. What am I doing? I think I did something wrong. I don't see any point in doing this 'cause I'm going to end up with a $Y^4$." She had not realized that she could tell her solution was incorrect by looking at the factored forms of the fractions. Although she had previously demonstrated she knew how to simplify fractions, she was well into the multiplication process before she abandoned her answer. The interviewer then guided Emily into setting up the related problem, "Find A such that $A \cdot B = C$." She solved the related problems and then solved the original problem. Then in checking her solution to her original problem she again inverted incorrectly.

Another error Emily made and quickly corrected may provide some evidence of her perception of the meaning of variable. It occurred in her solution to Task 1-15 (Create a division problem in which a trinomial divided by a binomial yields a binomial and a remainder of 1). In solving the task she first wrote $(X + Y)(2X + Y) = 2X^2 + XY + 2XY + Y^2 = 2X^2 + 3XY + Y^2$. At this point she decided that she wanted her solution to be in terms of only one variable so she substituted a 1 for the Y in the first binomial and a 2 for the Y in the second binomial. She continued to substitute the values into her intermediary product until she reached the $Y^2$ term. At that point she realized her error, dropped her attempted substitution, and went on to solve the problem with no difficulty.

These errors that Emily readily detected during the solution of a task may, after a period of time or in the midst of frustration, re-occur and go undetected. In her solution to Task 1-5 (Simplify: $(X + 1)^{1+n}\ (X + 1)^{1-n}$) she began, "When you multiply things raised to the power you - Gosh! What do you do? - you add the - I think - you add the exponents . . . I always get confused whether you multiply these or add them." Then she tried the simpler problem $X^5 \cdot X^3$ and explained why her solution to that problem had to be $X^8$ and solved the original problem. Weeks later her confusion with exponents reappeared in the midst of her frustration with multiplying radical expressions with different indices in Task 3-7 (What algebraic expression multiplied by $\sqrt[3]{2X}$ equals $\sqrt[12]{128X^{10}Y^6}$?). She initially attempted to apply negation to the task and wrote $\frac{\sqrt[12]{128X^{10}Y^6}}{\sqrt[3]{2X}}$. Next she multiplied numerator and denominator by $\sqrt[3]{4X}$. She commented, "Oh when you multiply roots . . . they have to be the same . . . No, not to multiply . . . Oh, if you multiply the twelfth root by the cube root . . . Can you multiply a cube root by a twelfth root?" The interviewer suggested trying a simpler case. After testing $\sqrt[3]{8} \cdot \sqrt{16}$, Emily noted, "They have to be the same root because if you multiply the two things you'll get the $\sqrt[6]{128}$, which isn't 8. And if you add them you get the fifth root and that still isn't 8. And if you subtract them you get the first root and that isn't 8 . . . . So I don't believe it can be done . . . . I don't see how you could multiply those unless you know what they are . . . You'd think it would have to be the fourth root of something 'cause if you could multiply them you'd get the twelfth root." At this point the interviewer suggested writing roots another way. Emily continued, "Well that's $(2X)^{1/3}$ and that's that whole thing raised to the 1/12 power. So it would make sense that this would be . . . $64X^5$ . . . . Now wait what am I doing?" She wrote: $(\quad)^{1/4}(2X)^{1/3} = (128X^{10}Y^6)^{\frac{1}{12}}$ and filled in the parentheses with $64X^5Y^6$. She questioned, "Is that right though? I mean it makes sense that you could do that but I don't think that's right." The following dialogue provided evidence of her false generalization with the properties of exponents in a frustrating task environment:

> Interviewer. You're going to multiply exponents?
>
> Emily. You don't multiply exponents! You add exponents! So what you're really trying to figure out is 1/3 plus what equals 1/12. So you change this into twelfths -- is 4/12 + (- 3/12) = 1/12. What did I just do? . . . Yeah, so you raise this to the -3/12 . . . I'm just guessing . . . and then you just multiply these and add the exponents and you should get that.
>
> Interviewer. When is it you add exponents?
>
> Emily. I don't know. If I knew I could do this. I can't remember any of that stuff. Okay, say you have $X \cdot X$ that equals $X^2$ so when you multiply you add exponents. When you add you have to have like exponents or else you can't add.
>
> Interviewer. OK, so if you take $Y^2 \cdot X^3$ do you add exponents?

> Emily. You can't multiply the $Y^2 \cdot X^3$. You just leave it . . . $X^3Y^2$ . . .
>
> Interviewer. Alright you wouldn't add the exponents in that case, would you. But you did here? What's the difference between those two?
>
> Emily. Yeah. The base is the same . . . So that's suppose to have significance . . . here my base is not the same I can not add the exponents. Then whatever in the world do I do?

The interviewer continued with very directed hints ("Those fractional exponents look intriguing") until the task was solved.

## Related Abilities, Attitudes, and Perceptions

Question 6. What evidence of the role of perception and memory on the solution of algebraic problems is provided by someone who is successful in a basic algebra course?

Question 7. What evidence does someone who is successful in a basic algebra course provide of her attitude towards algebra and her perception of her understanding of algebra?

### Perception and Memory of Algebraic Problems

Emily's solutions to tasks were often guided by the mental frame in which she placed them. In solving Task 1-6, part C (Simplify by finding the indicated product and combining similar terms: $(5X^n + 2)(6X^n + 3)$), for example, she wrote: $(5X^n + 2)(6X^n + 3) = 30X^{2n} + 15X^n + 12X^n + 6 = 30X^{2n} + 27X^n + 6 = 3(10X^{2n} + 9X^n + 2)$. After the interviewer asked Emily to look back at her answers, Emily noticed that she had factored part C and explained, "Usually when it says simplify you just go as far as you can." At the time of the interview she had just completed the chapter on factoring. Evidently she lost track of her goal and her perception of her final answer $30X^{2n} + 27X^{n} + 6$ immediately set off a factoring response. Her factoring reflex was not strong enough to include trying to factor $10X^{2n} + 9X^{n} + 2$.

Further evidence of the role of perception on Emily's solution was provided by her solution to Task 2-2 (Find the algebraic fraction in lowest terms which when multiplied by the sum of $\frac{3X}{5}$ and $\frac{X - 5}{15}$ yields 1). She began solving the task by adding $\frac{3X}{5}$ and $\frac{X - 5}{15}$. She wrote the sum as $\frac{9X + X - 5}{15}$ and divided one by the sum. Next she tried to simplify her solution by factoring the numerator and denominator. She was unable to simplify. She had perceived the $9X + X - 5$ as a quadratic polynomial, tried all combinations of factors of 9 and factors of 5, and declared the fraction to be in simplest terms. Afterwards the interviewer questioned where the $9X + X - 5$ came from. This was sufficient to help Emily break her mind set and complete her solution. She commented, "Oh what a dummy. I was thinking there was a square there. I'm just looking at things and not even seeing."

Emily was skillful at performing the algorithms of basic algebra. By applying negation she would set a chain of steps in motion that would yield the correct solution. At times however she was surprised at the results. For example, after solving Task 3-8 (Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$) she was surprised that her answer was 8: "I'm saying that 8 divided by $\sqrt{2}$ equals $4\sqrt{2}$. That doesn't sound right. Should I check it?" She multiplied $\sqrt{2}$ times $4\sqrt{2}$ and quickly verified that her solution was correct.

Other times Emily would catch bits of the statement of a task and quickly solve her incorrectly perceived problems. She needed to be guided back to the statement of Task 4-8 (Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ whose quotient is a positive integer) twice before she quickly solved the task. Her immediate perception was that the task was of the form $X + Y = 2$. She made two trials at substituting a complex number for Y and solving for X. Her first trial was unsuccessful since X was $-3i$ and did not meet the conditions of the problem. She felt her second trial was also unsuccessful since she substituted $2 + 3i$ for X in her equation and obtained $2 - 3i$ as the other complex number. She perceived that these two complex numbers were the same and therefore did not meet the conditions of the problem. At this point the interviewer suggested that Emily re-read the problem. Emily responded, "Whose quotient! That stinks . . . just ignore all that." This time she set up the equation $X \cdot Y = 4$. After a trial the interviewer hinted, "When it says quotient what do you do?" Now, Emily quickly solved the problem. Afterwards she commented, "I feel so stupid! I've done that before . . . I just take off and not even read what I'm . . . I read it, but it just doesn't register."

Attitudes and Self-Perceptions

Although Emily generally solved the Algebraic Interview Tasks very smoothly, she would consistently ask if her solution was correct. Almost as consistently the interviewer would respond with a line like, "Does it meet the conditions of the problem?" Emily would then check her solution, as if expecting that it would not work: "I don't think it's going to work. No, wait, yeah it is" (Task 3-6). This interchange between Emily and the interviewer occurred on tasks from each series and with each interviewer. The interviewer was generally viewed as the authority. Emily would look for re-assurance even for the checks. The following interchange characterized Emily's self-perception:

> Emily. So that's right?
>
> Interviewer. I think it's good enough.
>
> Emily. Good enough! I don't want to be good enough, I want to be right! (Task 3-4)

With regard to the style of presentation of the Algebra Interview Tasks, Emily indicated in her solution to Task 1-15 that she liked "creating" her own problems. She had little difficulty with most of the tasks.

The only specific topic or operation that Emily indicated any special feelings for was division and in particular, division of algebraic fractions. As she read the words "divided by" in Task 2-10 (Find the algebraic fraction in lowest terms if $\frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21}$ divided by it yields $\frac{2Y - 3}{Y - 7}$ ) she parenthetically added, "Oh yuk!." Again in her solution to Task 2-16 (Find the algebraic fraction in lowest terms which multiplied by $\frac{Y^2 - 7Y + 12}{Y^2 + Y - 2}$ yields $\frac{Y^2 - 3Y + 4}{Y - 1}$), when she realized that she "needed" to solve the problem by division, she complained, "Oh, I hate this." After completing the task she continued, "I hate division. I hate fractions worst of all."

## Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test

Emily's classifications for the 21 tasks of the Classification Test are presented in Table 4.2. She separated the cards into eight piles. The eight piles were described by Emily as: monomials, binomials to simplify (including radical expressions and complex numbers), trinomials "in the form of little word problems," simplifying fractions, "little word problems" with fractions, complex numbers and radical expressions in the form of "little word problems," factoring, and word problems without numbers. In her classification of tasks, she treated complex numbers and radical expressions with two terms as binomials for textbook style tasks. All other complex numbers and radical expression tasks were separated by Emily in terms of the style of presentation. She also

### Table 4.2

Emily's Solutions for the Classification Test and the Preference for Algebraic Problems Test Coded by
Type of Algebraic Expression - Polynomial (P), Algebraic Fraction (F), Radical (R), Complex Number (C);
Operation - Add (A), Subtract (S), Multiply (M), Divide (D), Factor (F); and
Style of Presentation - Textbook (T), One (1), Two (2), or Three (3) Missing Expressions to Find

#### CLASSIFICATION TEST

| PREFERENCE FOR ALGEBRAIC PROBLEMS TEST | SIMPLIFY FRACTION | FRACTION TO FIND | RADICAL | SIMPLIFY | WORD PROBLEM | POLYNOMIAL TO FIND | FACTOR | MONOMIAL | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | P-M-3 | | 1 |
| 2 | | | | | | | P-F-T | | 2 |
| 3 | | | | | | | P-M-2 | | 3 |
| 4 | | | | P-S-T | | | | | 4 |
| 5 | | | | P-M-T | | | | | 5 |
| 6 | | | | C-M-T | | | | | 6 |
| 7 | | | | | | P-S-1 | | | 7 |
| 8 | | | R-M-1 | | | | | | 8 |
| 9 | F-S-2 | | | | | | | | 9 |
| 10 | | | R-A-1 | | | | | | 10 |
| 11 | | F-A-1 | | | | | | | 11 |
| 12 | | | C-S-1 | | | | | | 12 |
| 13 | | | | | C-M-3 | | | | 13 |
| 14 | | | R-D-1 | | | | | | 14 |
| 15 | | F-D-1 | | | | | | | 15 |
| 16 | F-D-T | | | | | | | | 16 |
| 17 | | | | C-A-T | | | | | 17 |
| 18 | F-S-T | | | | | | | | 18 |
| 19 | | | | R-A-T | | | | | 19 |
| 20 | | | | | | | | P-D-2 | 20 |
| 21 | R-D-T | | | | | | | | 21 |

categorized based on the number of terms of a polynomial or whether the task involved fractions.

Emily experienced some frustration with the Mathematical Memory Test. Without looking at the cards, she was able to describe only five of her eight categories. The card with the circled number from the pile of "fractions to be simplified" was the task - "Simplify: $\frac{A}{A^2 - 16} - \frac{A + 1}{A^2 - 5A + 4}$." She recalled: "It said simplify. It was a fraction. It had the number A in it. It was one fraction equals another fraction. I think. Maybe it was one fraction added to another. I know it was simplifying. There was two different fractions. A was capitalized. You didn't tell me I was supposed to remember stuff."

Emily recalled that another card said:

> The product was a negative integer. I guess it was two binomials whose product was a negative integer. And you were supposed to find the binomial and find the integer too. I might have it mixed up. There was another one that gave two binomials and said that they equal a binomial when multiplied together. I might have it mixed up with that one. But I don't know if that was the top card. So that's no help. And you were supposed to find both the binomials and the product.

The first of these recalled cards probably refers to the circled task from the "word problem without numbers" pile, "Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$, such that their product is a negative integer." If so, this provides another instance in which complex numbers were recalled as binomials. The second of the recalled cards was correctly stated and was not circled. It was included in the "factoring" pile. Although she recalled the cards together on the Mathematical Memory Task, she did not include the complex number task with her pile of factoring tasks.

Emily recalled that the factoring task that was circled had "X's" in it, but did not know if she could factor it. She did volunteer that she could factor the one where you find two binomials because "it's one of those rules that you memorize." She continued, "that's why I remembered it. Because I thought, 'Oh that's easy' as I flipped thru."

After the cards were reshuffled, Emily arranged the tasks in the order in which she would like to solve the problems. Her order of preference is presented in Table 4.2. She spent nearly 8 minutes to order the tasks and noted, "It's hard to decide when you really don't want to do any of them."

Emily put the factoring problems first because she liked doing them. She described her procedure as follows: "I tried to see what I already knew how to do without having to work too much and then put the ones where I'd have to think about it and work a little more on it toward the end."

Emily's order of preference did not follow the order that instruction normally follows. She commented, "I like multiplication and adding a lot better than subtracting or dividing." Six of the first eight tasks in order of preference involved multiplication (or factoring). The only task involving multiplication that she ordered later than eighth place was the "word problem without numbers." Next she chose the tasks based on style of presentation. "I like it when they give you the answer and let you find the other one, rather than like most problems, you know, they give you the two things and you have to find the answer. Which is not hard to do. But I just kind of like it twisted around backwards. So I put all of these together." Tasks 7 thru 14 were drawn from the algebraic interview tasks. She placed the task - "Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$" - ninth in her order of preference. As she explained, "This is one where you just make up one of the fractions and then you just find the other one so it's not all that hard." When asked if she remembered doing the task during a previous interview she replied, "Yeah, I did. Cause I got all messed up on it and then I said wait I'm making this too hard. Cause when you first look at it you think 'Gosh, how in the world am I going to do that!' and I scribbled all over my paper. And then it just hit me that it wasn't that big of a deal."

Emily expressed some concern over her placement of the task - "Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$." She stated, "This one I couldn't decide what to do with this, cause I never can remember the rules for adding things under a radical. So I just stuck it there." She did not recall ever having attempted to solve this task before. Even though she had solved the fractions task weeks earlier than the radical expressions task and had spent nearly 50 percent more time on the radical expressions task, she recalled her solution process for the fractions task, but did not even remember attempting the radical expressions task. "Did I solve it before? I can't remember. I'd like to forget about square roots."

Emily ordered the division tasks further back in her ordering. As she explained for the task - "Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$." "This one I moved farther back because it's division, which is the worse! But it's really multiplication. But anyway it said divided, so that just blew its chances of being in the front. And it's also square root. But I don't think it would be that hard." She explained her feelings about division as follows: "When I was in the fourth grade they skipped me a year in math and I missed division so I had to pick it up on my own. And it was rough and I hated it. So I've always hated it. I just don't like it. Cause like I'd come home from school and my mother would sit down with the multiplication tables trying to teach me to do it backwards. And it was just a battle every day after school and I hated it. So I never have liked division." She ordered the five tasks involving an indicated division among the last eight tasks of her list.

Emily held off five of the nine "textbook" tasks until the last five positions in her order of preference. Besides division tasks the "least liked" tasks included fractions, radical expressions, and monomials. She placed the monomial task - "Find two monomials whose quotient is $-5AB^2C$" - twentieth in her list. She explained, "This one, I never have liked this one from the very first time I saw it. It was in a group all by itself even today. I don't know why that's so hard. Quotient that means dividing - I guess that's why. But really, don't you just kind of factor that -- in a way. But factoring when there's no adding signs that messes in. I don't know I just didn't like that one." Although she reacted negatively to this item she had solved it with little difficulty during her second interview session weeks earlier.

## Summary

Emily was characterized by her emphasis on negation as her general method for solving the algebraic tasks. She would quickly apply negation to solve the tasks. Emily also used negation on interview tasks with more than one algebraic expression to find.

It was difficult for Emily to think of compensation as a viable approach for solving the tasks. Even with interviewer interference, Emily reluctantly applied compensation. The only tasks in which she consistently applied compensation were tasks involving the multiplication of polynomials.

On tasks Emily solved by negation, she generalized her solutions to other examples by choosing a different algebraic expression for one of the unknown expressions. On tasks Emily solved by compensation, she was a bit unsure of how to generate other solutions. Emily demonstrated an ability to transfer processes and operations from polynomials, radical expressions, and equations, to her work with complex numbers.

The errors Emily made reflect her strong reliance on rules for guidance in solving algebraic tasks. Her solutions to tasks were guided by the mental frame in which she placed them. Although she was skillful at performing the algorithms of basic algebra, she lacked confidence in her solutions.

## Lynn

### Reversibility and Flexibility

Question 1. What evidence of a reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

Question 2. What evidence of a flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

### A Pattern of Behavior

The main approach used by Lynn on the Algebra Interview Tasks was compensation (see Table 4.3). She applied compensation to each of the 12 polynomial expressions tasks for which reversibility was necessary to solve the task. In

Table 4.3

Algebraic Interview Tasks Solved by Lynn With Hints (Underlined) or Without Hints (Not Underlined)

by Negation (N), Compensation (C), or Another (A) Approach

(The number in parentheses indicates the order of presentation for the task.)

| | Find the Missing Algebraic Expression: | | | Vocabulary and Algebraic Symbolism |
|---|---|---|---|---|
| | One Expression<br>□ ∗ B = C or A ∗ △ = C | Two Expressions<br>□ ∗ △ = C | Three Expressions<br>□ ∗ △ = ○ | |
| Polynomials | − 14(14,C$^{b}$)<br>× 12(12,C); $\underline{13}$(13,C$^{b}$) | − 8(2,C)<br>× 9(9,C); 10(10,C)<br>÷ 7(8,C) | − 2(4,C)<br>× 4(1,C); 11(11,C)<br>÷ 15(15,C) | 1(3,A); 3(5,A)<br>5(6,A); 6(7,A)<br>16(16,A) |
| Algebraic Fractions | + $\underline{1}$(17,C); $\underline{5}$(21,C$^{c}$)<br>− 4(20,C); 15(22,N)<br>× $\underline{2}$(18,C); $\underline{6}$(22,C$^{d}$); $\underline{16}$(23,N$^{a}$)<br>÷ $\underline{3}$(19,N); $\underline{10}$(24,C$^{a}$) | − 12(25,C); 13(26,C$^{d}$ | | |
| Radical Expressions | + 4(30,N$^{a}$)<br>− 9(34,C)<br>× 3(29,C); 5(31,C); 6(32,N);<br>$\underline{7}$(33,N); $\underline{11}$(43,N)<br>÷ 8(33,C); 10(35,C$^{d}$) | | | 1(27,A)<br>2(28,A$^{d}$) |
| Complex Numbers | + 1(36,C)<br>− 4(40,C)<br>× 2(37,C); 3(41,N) | + 7(42,C)<br>− 5(38,C) | × 6(39,C)<br>÷ $\underline{8}$(43,N) | |

[a] First attempted compensation.
[b] First attempted compensation, then negation, then compensation.
[c] First attempted negation.
[d] Never solved correctly.

two of these instances she originally attempted to apply negation, but switched her approach to compensation to complete her solution. One example of her application of a compensation approach is provided by her solution to Task 1-15 (Create a division problem in which a trinomial divided by a binomial yields a binomial and a remainder of 1). First she arbitrarily chose X + 3 as the binomial and X as the first term of the trinomial. Next, she wrote her problem in the form of a long division problem and as if performing a previously learned algorithm; stated the following steps:

1. X times X + 3 is $X^2 + 3X$
2. Okay, I could use 5X
3. That would make this 2X
4. Then this would be 2X + 6
5. And I could make this a -7 . . .
   No, wait subtract . . .
   Bring down the -7 . . .
   No, make this a +7
6. And when you subtract it you'll
   have a +1. Ha! Ha! Okay.

The initial step of
Lynn's written work:

$$\begin{array}{r} X \\ X + 3\;\overline{)\;X^2\phantom{+3X}} \\ \underline{X^2 + 3X} \end{array}$$

After completing the process Lynn checked her solution by actually dividing $X^2 + 5X + 7$ by X + 3.

Similarly, Task 1-12 (What trinomial multiplied by 4A + Y yields the product $64A^3 + Y^3$?) Lynn determined the factors of the requested trinomial by writing "(4A + Y)(     )" and then filling in the second pair of parentheses with the terms necessary to obtain a product of $64A^3 + Y^3$.

Task 1-13 (What trinomial multiplied by A - B - C yields the product $A^2 - B^2 - C^2 - 2BC$?) was the first task that Lynn initially attempted to solve by negation (dividing

$A - B - C$ into $A^2 - B^2 - C^2 - 2BC$). She was quickly frustrated by being unsure of how to perform the indicated division: "This is so simple, I think, but I can't figure it out. I'm not really thinking anything. Let me do a different one. Give me a hint . . . . Can you give me a hint?" The interviewer suggested thinking about a numerical problem of the same form, but Lynn solved the numerical problem by negation (division) and so was not aided by the hint. Once again she attempted to divide $A - B - C$ into $A^2 - B^2 - C^2 - 2BC$. Next the interviewer suggested a similar problem involving polynomials: "What binomial multiplied by $X + 2$ equals $X^2 + 5X + 6$?" Lynn quickly stated the solution to the related task and returned to Task 1-13: "Okay you could factor that [a form of compensation] . . . But I can't see . . . How would we factor it. I'm trying to figure out how to factor. It's not a square or anything like that. So . . . . you could group two of them together. But that won't work 'cause you can't get a common factor." Unlike her solution to Task 1-12, she did not attempt to utilize the given factor in order to find the missing one. Again she switched her approach to negation. After she divided A into $A^2$ and multiplied the A times $A - B - C$, she complained, "I'm stuck with an AB and I don't know where I want to put that." At this point the interviewer asked, "What do you think your answer will look like." Lynn replied, "A trinomial . . . Oh! Wait a second . . . $A - B - C$ . . . I think I've got it." Again the interviewer's hint prompted a shift in Lynn's approach to her solution. She began multiplying the A - B - C times trinomials of the form A * B o C where * and o are either addition or subtraction. This form of compensation was led by her desire to make "the middle terms cancel out . . . subtract out." By changing the signs, she arrived at a solution to the task. Afterwards she again attempted to solve the task by long division. She got as far as having A + B in her quotient and then stopped saying, "I guess it would work."

Lynn's dependence on compensation was also very evident in her completion of the Algebraic Fractions Series. She used a compensation approach in 8 of the 10 tasks attempted (see Table 4.3). She was uncomfortable with the use of negation -- if the given operation was addition she was uneasy using subtraction to solve the task. She was not sure if an answer obtained that way was "really" correct. For example, on Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$), she re-read the problem several times. She hypothesized that the denominator of the missing fraction was X - 1. Since she was unable to visually determine the numerator she tabled her procedure and applied negation to the task: "If I have $\frac{2X - 15}{X^2 - 6X + 5}$ and then I want to find the number which when added to $\frac{X}{X - 5}$ yields that . . . if I subtract $\frac{X}{X - 5}$ from that [$\frac{2X - 15}{X^2 - 6X + 5}$] that will work that way." She paused for a while and then continued, "I was just thinking if I add I can do the opposite and subtract and still it can all come out somehow . . . . Something's coming out but I don't think it's the answer! I think the denominator needs to be X - 1." By this point she had obtained $2X + 5 - X^2 + X$ as the numerator of the fraction and decided to drop her approach because her answer did not look right. Since the solution she was obtaining did not meet her hypothesized condition (a denominator of X - 1), she was prepared to go to a new problem. At this point the interviewer wrote the following on Lynn's paper:

$$\frac{X^2 - X}{X^2 - 6X + 5} + \qquad\qquad = \frac{2X - 15}{X^2 - 6X + 5}$$

Lynn quickly exclaimed, "Oh! Oh! Now I see! Ok it would be . . . $-X^2 + 3X - 15$ [writes $\frac{-X^2 + 3X - 15}{X^2 - 6X + 5}$]. Oh! I had it right there!"

Lynn also tended to apply compensation in her solutions for the Radical Expressions and Complex Number series. She applied compensation to solve 12 of the 13 radical expression tasks and complex number tasks in which reversibility was necessary for solution (see Table 4.3). Her application of compensation to the radical expressions task was exemplified in her solution to Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals -14?). She began her solution by setting up the statement:

$$(2 + 3\sqrt{2}) \qquad\qquad = -14$$

Then she tested different radical expressions in an effort to find one that multiplied by $2 + 3\sqrt{2}$ would "get rid of the radical." She first tested multiplying by " - $\sqrt{8}$." Since she still had a radical, she tested 2 - 3$\sqrt{2}$.

Lynn's "test, evaluate, and re-test" technique was also evident in her solutions to the complex number tasks. Her solution to Task 4-6 (Find two complex numbers of the form A + Bi where A $\neq$ 0 and B $\neq$ 0, such that their product is a negative integer) provided evidence of the "homing in" quality of her compensation technique. First, she multiplied (2 + 3i)(2 - 3i). Next she tried (2 - 3i)(2 - 3i). She paused and then said, "When it says negative integer it can't have an i in it can it? So I try 2 - . . . No, I don't want 2 . . . (3 - 4i) times . . . . What number should I use this time? No wait a minute let me go back to this one. I have (2 + 3i)(2 - 3i) . . . and that was 13. Okay, if you make this (6 - 4i)(6 + 4i) . . . ." Then she tested (-6 - 4i)(6 + i), (6 + 4i)(6 + 4i) and finally (-6 + 4i)(6 + 4i).

Flexibility of Approach

Lynn rarely applied negation to tasks and when she did she frequently confused her operations. For instance, after solving Task 4-4 (Find the complex number which subtracted from 5 + 3i equals 7 - 3i) by compensation, she was asked if she could have solved the problem another way. The following is an excerpt from the transcript of her solution to Task 4-4.

> *Lynn*. You could have subtracted 7 - 3i - (5 + 3i).
>
> *Interviewer*. Would that have worked?
>
> *Lynn*. Yeah, 2 - 6i [her original solution was -2 + 6i]
>
> *Interviewer*. So either way you could have gotten . . .
>
> *Lynn*. Yeah . . . No . . . Yeah, You would.
>
> *Interviewer*. [After a long pause] No, wait I'm confused. Wasn't this a -2 up here?
>
> *Lynn*. Yeah, but it was positive right here. What is the complex number that subtracted from 5 + 3i equals 7 - 3i? Oh! Oh! OK, wait a minute! Oh, you'd subtract . . . [subtract 5 + 3i - (7 - 3i)]. Oh, ok I had them backwards. Then it's the same.

The only task from the Radical Expressions and Complex Number Series that Lynn solved strictly by negation was Task 4-3 (Find the complex number which multiplied by 5 + 3i equals 13 + i). After completing the task she was asked why she had not tried the same approach to solve this tasks that she had used on the others in the series, namely set up the skeleton:

$$(5 + 3i) \qquad\qquad = -13 + i.$$

Lynn responded: "I didn't even think about that. I don't know what I was thinking about. If I did it that way I probably would have got it."

### Reversibility of Multiplication and Factoring

Lynn utilized the relationship between factoring and multiplying polynomials on all but one of the tasks involving the multiplication of polynomials. In addition to her solution to Task 1-12, she used her ability to factor (a form of compensation) to solve Tasks 1-4, 1-9, 1-11, and 1-16. The only task involving the multiplication of polynomials in which she did not use her ability to factor polynomials was Task 1-13. After solving Task 1-16 (Find the product of $2A + B$ and $3A^2B + AB - C$. Factor the product), she was asked what she felt the purpose of the task was? She responded, "To see if when you got the answer, if you figured out that when you factored you get what you started with."

Reversibility of Multistep Algorithms

On tasks involving multistep algorithms, where Lynn was unsure how to apply compensation, she would re-read the problem many times. Task 2-3 (Find the algebraic fraction in lowest terms which divided by $\frac{5X - 1}{X^2 - 25}$ yields the quotient $-2$) was re-read nine times before she lamented, "Oh, I don't even know where to start. I know you've got to find the other number, but I don't know how to find it." She re-read the problem a few more times. She tried to solve the problem mentally and wrote nothing on her paper to relate the problem to. At this point, the interviewer asked, "Suppose you did have the answer to the problem, how would you check it to see if you were right?" Lynn answered, "Multiply the answer times the divisor." Then she gave two numerical examples. She used negation to check her solutions, but did not use it to solve the problem. After a long pause, she continued, "Okay they . . . You could multiply . . . Could you do the same thing with this one? Could you multiply -2 times the number here? . . . I think so . . . $\frac{-2(5X - 1)}{X^2 - 25}$ times . . . Oh you're dividing, why do I keep multiplying? This is making me mad that I can't figure this out! I keep wanting to multiply instead of divide."

This difficulty in mentally reversing the steps of a multistep algorithm was not limited to Lynn's use of negation. She also had difficulty applying compensation to multistep algorithms. On Task 2-1 (Find the algebraic fraction in lowest terms which when added to the sum of $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ yields $\frac{3X + Y}{Y}$), she fought with the desire to "work forwards" through the problem and to accept the visually balanced solution $\frac{2X + Y}{X}$. After reading the problem several times she added $\frac{3X + Y}{Y} + \frac{2X + Y}{X}$ and then broke from her solution process:

> I keep wanting to say that the answer is $\frac{2X + Y}{X}$, but I don't think that's right . . . . When added to the sum . . . when added to the sum . . . . Well the sum of this . . . I don't know . . . this is part of . . . the $\frac{3X + Y}{Y}$ is . . . . OK, you've got to find a fraction that when added to the sum of $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ yields $\frac{3X + Y}{Y}$. If . . . Gee! I don't know how to say this . . . Oh . . . I don't know . . . if you add these then this has got to be the other . . . . Do you know what I'm saying? Oh, if you add these two and you want to find the fraction . . . which when added to the sum . . . I don't know . . . which when added to the sum . . . . I think it's $\frac{2X + Y}{X}$. Is that right?

After Lynn checked her solution she quickly switched to $\frac{-2X - Y}{X}$ as the answer to the problem.

The fewer the visual cues available, the greater the difficulty that Lynn experienced with the tasks. Her solution to Task 2-13 (Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$) was a frustrated series of trials in which she had difficulty coordinating the conditions of the task. Despite an attempt by the interviewer to guide Lynn to thinking of the intermediary step of reducing a fraction to obtain $\frac{2A}{B}$, she was unable to solve the problem.

Lynn's difficulty with reversing multistep algorithms also increased in areas in which she felt less competent, for example working with radical expressions. She had difficulty in coordinating the conditions necessary for using compensation to solve Task 3-10 (Find the real number which divided by $\sqrt{3}$ equals $3 - \sqrt{3}$. As she set up a skeleton indicating the division as a fraction, she indicated that she had to get rid of the $\sqrt{3}$ in the denominator. She initially tried $3 - 2\sqrt{3}$ as the numerator for her fraction, but dropped it because she realized that she had to divide by $\sqrt{3}$. Next she tried $3\sqrt{3} - \sqrt{3}$ and then $\sqrt{9} - \sqrt{3}$. In each of these trials she was focusing on the need to have 3 as the first term of her solution. It was not until several minutes later that she commented, "If you multiply $(3 - \sqrt{3})\sqrt{3}$ you get $3\sqrt{3} - \sqrt{9} = 3\sqrt{3} - 3 = 3(\sqrt{3} - 1)$" and then rationalized the denominator of $\frac{3\sqrt{3} - \sqrt{9}}{\sqrt{3}}$ to show that she would get $3 - \sqrt{3}$.

## Ability to Generalize and Transfer

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

Question 5. What evidence of the use of false generalizations and false transfers is provided by someone who is successful in a basic algebra course?

### Ability to Deduce the General from the Specific

Lynn was generally able to distinguish between tasks with unique solution and tasks with more than one solution. She provided no evidence of an ability to state her solutions in general terms. On most of the Polynomial Series Tasks with more than one solution she provided another solution if requested. She did not volunteer another solution if one was not specifically requested and did not volunteer a generalized solution.

The only instance in which Lynn indicated some confusion with regard to the possibility of multiple solutions for a task occurred in her solution to Task 1-7 (Find two monomials whose quotient is $-5AB^2C$). She quickly solved the task and was asked by the interviewer if she felt that her answer was unique. She responded, "No . . . .

Yeah, I guess it is pretty unique." The interviewer pointed out that Lynn could have used 30 and -6 and she responded, "Except for numbers." After a pause, the interviewer noted that Lynn could use $A^5$ and $A^4$ and after another pause the interviewer continued that Lynn could put a D in both monomials. She did not volunteer any other possibilities and merely acknowledged, "I guess it's not unique."

On Task 2-12 (Create a subtraction problem involving two algebraic fractions in lowest terms with the _same_ denominator and whose difference is $\frac{2A}{B}$), Lynn readily generalized her solution to fractions that have "any numbers that when they subtract equal 2 . . . like $\frac{5A}{B} - \frac{3A}{B}$." She stated that the denominators of the fractions would have to be B "cause you couldn't get a common denominator and have other letters in there."

### Ability to Generalize from Task to Task

Lynn demonstrated her ability to generalize processes from task to task in her solution to Task 1-10 (Factor the following binomial into a product of a monomial and a binomial: $A^{m+3} + A^3$). After she stated that she was unable to think of a possible solution, the interviewer suggested that she try to solve the following problem instead: "Factor $A^5 + A^3$ into a product of a monomial and a binomial. She solved the new problem and when requested gave two other solutions for it. After obtaining the solution $(A^2 + 1)A^3$, she stated, "Oh, I think I've got it. It's sort of like this with the 1 and I've got to figure out where the 1 goes . . . $A^3(A^m + 1)$. Okay, I've got it." When requested, she gave two other solutions to Task 1-10.

Further evidence of Lynn's ability to generalize her solution process was provided by her solution to Task 2-6 (Find the algebraic fraction in lowest terms which multiplied by $\frac{X - 2}{X^2}$ yields $\frac{X - 1}{X^2 + X}$). In solving this task she generalized the procedure that she used to solve Task 2-2 (Find the algebraic fraction in lowest terms which when multiplied by the sum of $\frac{3X}{5}$ and $\frac{X - 5}{15}$ yields 1). This procedure involved multiplying by the multiplicative inverse of the sum of $\frac{3X}{5} + \frac{X - 5}{15}$. She applied the procedure to Task 2-6 by first multiplying $\frac{X - 2}{X^2}$ by its multiplicative inverse to get 1 and then multiplying by $\frac{X - 1}{X^2 + X}$ to obtain the required product. As a single fraction she stated her answer as $\frac{X^2(X - 1)}{(X - 2)(X^2 + X)}$. She was unable to complete the task because in writing her answer she wrote instead $\frac{X\,(X - 1)}{-2(X^2 + X)}$ and was frustrated when she tried to check if it met the conditions of the problem. She became confused and suggested putting the problem aside for a later time: "If I were doing this at home I'd put it down and come back in an hour." She did not generalize this process to all multiplication tasks of the Algebraic Fractions Series.

### Ability to Transfer

Lynn did not tend to use arithmetic problems to help her solve algebraic problems unless she was directed to by the interviewer. Lynn's ability to operate with arithmetic problems did not always transfer to her work with polynomials. Although she would follow the direction of thought suggested by the arithmetic variant she was not convinced that the transferred procedure would or could transfer. These suggestions occurred at points in a solution in which she indicated that she needed a hint. Her difficulty with the application of negation in general and the application of compensation in multistep algorithms generally preceded or prompted the interviewer's suggestion to try the task with numbers.

On Task 2-13 (Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$), the interviewer's suggestion to try the task with numbers met with limited success. She chose 3/4 as her answer and quickly stated that the fractions could be 5/4 and 1/2 (by compensation). Next she tried to no avail to apply her solution to the arithmetic problem to her algebraic task. This time the interviewer suggested that Lynn use 10/13 as the answer. She was unable to solve the task. The interviewer switched again and chose 6/5 as the answer for a related arithmetic variant of the task. In a very directed fashion the interviewer led Lynn to a solution for the arithmetic problem. She was unable to transfer the technique to Task 2-13, complained of a headache, and asked to stop working on the problem.

Lynn felt very unsure of her ability to solve problems dealing with complex numbers. In her solution to three complex number tasks -- Task 4-1, Task 4-4, and Task 4-10 -- she needed to be reminded of the relationship between the addition of polynomials and the addition of complex numbers. Once the relationship was recognized she easily transferred her solution process for addition and subtraction of polynomials to complex numbers. The following discussion between Lynn and the interviewer took place during Lynn's solution to Task 4-1 (Find the complex number which added to -3 + 4i equals 2 + i):

> Lynn. Find the . . . Oh God! . . . I don't even know how you do these things . . . which added to -3 + 4i = 2 + i . . . I can't even remember how to do these complex numbers. Okay . . . How do you add complex numbers? . . . I can't even remember this . . . Alright, if you had 4i + 2 . . . I remember how to multiply but I can't remember how to add them.
>
> Interviewer. You treat them very much the same way that you do algebraic expressions.
>
> Lynn. Oh, is it like the first times the second . . .the third times the fourth . . .times . . .
>
> Interviewer. Well, for adding it's even simpler than that.
>
> Lynn. Oh, do you just add the real parts and then add the . . . Okay.

Lynn quickly solved the task by compensation.

During the next interview session Lynn was asked to solve Task 4-4 (Find the complex number which subtracted from 5 + 3i equals 7 - 3i). Her first reaction to the task was "Wait. How do you subtract! I hate complex numbers! I can't even remember how to add them! Oh, I remember now." She solved the task in the same way in which she had previously solved Task 1-14 (What trinomial subtracted from $5X^2Y - XY^2 + 7$ yields the difference $-X^2Y + 8$?). She filled in values that would make the statement true (by compensation). After solving the task she compared it to her work with polynomials: "It's sort of like, you know, X + Y, that kind of stuff. It's just confusing."

Later in that same interview session Lynn was asked to solve Task 4-7 (Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$ such that the sum of the two complex numbers is i). Once again she initially had difficulty recalling how to add complex numbers:

> Okay, you just add normal . . .[she writes (-4  ) + (4  )]. How do you add complex numbers? . . . [she writes off to the side of the paper, (2 + 3i) + (2 + 3i) = 4 + 6i] . . . Oh, you just add . . . How do you add complex numbers? I did this just two problems ago and I can't even remember . . . Oh, you just add normal don't you . . . Two complex numbers and the answer is i . . . Oh! I see! Make this 3 and this 2 . . . So if you added (-4 + 3i) + (4 - 2i) the fours would add out and you'd be left with an i.

Lynn experienced a similar confusion in her efforts to transfer her processes for solving tasks involving algebraic expressions and radicals to tasks involving complex numbers. At one point in her solution to Task 4-3 (Find the complex number which multiplied by 5 + 3i equals 13 + i) she was trying to rationalize the denominator of the fraction $\frac{13 + i}{5 + 3i}$. First she tried multiplying 5 + 3i. She multiplied (5 + 3i)(5 + 3i) using an incomplete version of a formula she had originally learned in her basic algebra course. With help from the interviewer she determined that multiplying by $5 + 3i$ would not rationalize the denominator. Next she tried multiplying by $3i$. Before multiplying she questioned, "Would it be the same if I put this [i] into a radical and tried to get rid of it? . . . I don't know. I don't know how to do this." With guidance from the interviewer Lynn was able to concentrate on her goal of having no "i's" in the denominator: "Oh, I got it! I got it! It's $(5 + 3i)(5 - 3i) = 25 - 15i + 15i - 9i^2 = 25 + 9 = 34$. Yeah, that would work." She continued her solution process, quickly multiplying the numerator by $5 - 3i$ and correctly "factoring out" a 34. Afterwards she was asked what the trick to the problem was. She replied, "Trying to get rid of the i in the denominator. I knew you had to get rid of the middle term so for some reason I just figured out . . . you've got to multiply the sum and difference."

Lynn experienced a feeling of "lack of closure" in her solution to Task 3-4 (Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$). In an effort to obtain closure for the task she attempted to transfer many processes in an effort to complete the task. In each case she abandoned those procedures that would have been falsely transferred to the addition of radicals. Her initial trials included $\sqrt{2} + \sqrt{3} = \sqrt{6}$ and $\sqrt{3} + \sqrt{3} = 6$. Then she began to try to change the radicals into an equivalent form:

> I keep wanting to make the $\sqrt{3}$ into something else . . . like make it $\sqrt{12}$ and when you take the square root of that you have $2\sqrt{3}$. Except I keep wanting to do that and make it into 6 or something . . . $\sqrt{24}$ is $2\sqrt{6}$ and now I keep wanting to say that's equal to the $\sqrt{3}$. Could I change it like you do with fractions to get a common denominator? I think of it like fractions or something, like something + 1/2 = 3/4 and trying to get a common thing and trying to make all these radicals 6, but still equal $\sqrt{3}$.

The interviewer attempted to help Lynn achieve closure by asking her to solve the related problem, "What would you add to A to get B?" She replied, "Well subtract B - A. But I don't know how you'd subtract B - A . . . so if A was 5 and B was 2 that would give me 3; but that doesn't help me here. Can you multiply, like if it was $\sqrt{6} - \sqrt{3}$? Can you just multiply by itself and not do anything to the $\sqrt{6}$? . . . $\sqrt{6} - \sqrt{3}$ is like the B - A." She was not comfortable leaving that as her solution, but despite her feeling of "lack of closure" she was unwilling to apply a false generalization or false transfer to resolve the conflict. The next section describes those instances in which she did apply false generalizations or false transfers.

## Errors, False Generalizations, and False Transfers

Any errors made by Lynn on the tasks in the Polynomial Series tended to be a product of the speed with which she attempted to solve each task. The errors such as stating that $(X + 1)^2 = X^2 + 2X + 2$ were made in haste and quickly corrected by Lynn with and without interviewer assistance. These errors were more like typos, made without thought, than false generalizations. For example, in solving Task 1-4 (Find two binomials whose product is a binomial. Find the product) she stated that the binomials could be $2X - Y$ and $2X + Y$ and that their product was $9X^2 - Y^2$. Like a typist hitting the wrong key, she apparently pulled from memory the incorrect square number. When she looked back at her work, she corrected her error.

Lynn's errors increased in number on the tasks of the Algebraic Fractions Series. The errors made could still be classified as typos, . . . for example, writing "-2" instead of "X - 2" while stating the latter. Errors of this type were made on five of the tasks from the Algebraic Fractions Series.

Evidence of the confusion and frustration these errors added to Lynn's solution process was provided in her solution to Task 2-16 (Find the algebraic fraction in lowest terms which multiplied by $\frac{Y^2 - 7Y + 12}{Y^2 + Y - 2}$ yields $\frac{Y^2 - 3Y + 4}{Y - 1}$). She initially factored the $Y^2 - 3Y + 4$ incorrectly as $(Y - 3)(Y - 1)$. This error was carried throughout her solution. Later in her solution she applied negation incorrectly:

> This is my final answer and this is one of the numbers that I multiply. I'm thinking if there's a way that I can divide somehow, maybe, and get the other term, but I don't know how . . .. I want to divide $Y^2 - 7Y + 12$ . . . that factors into $\frac{(Y - 4)(Y - 3)}{Y^2 + Y - 2}$ divided by -- times $\frac{Y - 3}{(Y - 4)(Y - 1)}$.

Lynn multiplied and obtained $\frac{Y - 4}{Y^2 + Y - 2}$. Lynn's division included two additional errors: dividing by the wrong fraction and not factoring $Y^2 + Y - 2$ (she stated later in the solution that it could not be factored). The errors she made added to her confusion about the appropriateness of negation for solving the tasks. In frustration she began multiplying the polynomials of the original numerators and the original denominators, just trying to find something to try. Eventually she asked to put the problem aside.

Lynn had difficulty recalling a formula she had been given for multiplying complex numbers. This difficulty continued from task to task. This difficulty first arose in her solution to Task 4-2 (Find the complex number which multiplied by 3 - i equals 10):

> Oh, we had a . . . she gave us a little form to follow like AB - whatever . . . and I better remember what it is . . . I think it's AC - BD times . . . like if you had (4 + 3i)(2 + 2i). Oh, you do first and third minus BD and then you have AD + BC. [Lynn writes ( )(3 - i) = 10.] So you want to find the number multiplied by 3 - i that equals 10, which means that you don't have an i. So if A = 3 . . . (3 + i)(3 - i) would give you $9 + i^2$ . . . . Can I ask you a question? Is this the right way to multiply?

Lynn had written the product of (A + Bi)(C + Di) as (AC - BD)(AD + BC). Since she felt that the B and D included the i, she obtained (9 - 1)(-3 + 3) = 8(0) as her product. Although she had transferred the process of multiplying by the sum and difference, based on her formula, she believed that it was incorrect. After she tried (3 + i)(3 + i) the interviewer corrected the formula by pointing out that the B and D are the coefficients of the complex terms and that the proper formula was

$(A + Bi)(C + Di) = (AC - BD) + (AD + BC)i$. With guidance from the interviewer Lynn completed her solution for the task.

Later in that session Lynn tried to recall the formula to help her solve Task 4-6 (Find two complex numbers of the form $A + Bi$ when $A \neq 0$ and $B \neq 0$, such that their product is a negative integer). This time she wrote the following expression as the product of $A + Bi$ and $C + Di$:

$$(AB - CD)(AD + BC).$$

Once again the interviewer corrected her formula and Lynn proceeded with her solutions to the task.

During the next session Lynn incorrectly stated the formula for multiplying complex numbers for a third time. In attempting to solve Task 4-3 (Find the complex number which multiplied by $5 + 3i$ equals $13 + i$) she began:

> Now how do you multiply complex numbers? I remember we had one of these last week and I had to go through this figuring how to multiply complex numbers. You know we had a formula in math with A's and B's . . . things that you could find . . . certain numbers stood for A . . . whatever. Now I can't remember which they were AB and CD . . . I think it was (AC - BD) + (AD + BC) . . . I think that's what it is . . . No, this is . . . No, I think that's right.

Lynn indicated the solution as $\frac{13 + i}{5 + 3i}$ and tried to rationalize the denominator by multiplying by $5 + 3i$. She wrote, "$(5 + 3i)(5 + 3i) = (25 - 9i^2) + (15i + 15i)$." In the following excerpt from her solution to Task 4-3 the interviewer attempted to guide Lynn to the appropriate formula:

Interviewer. Why is that $-9i^2$?

Lynn. Well when you multiply you've got to multiply the first terms and then it's minus these two . . . the i terms.

Interviewer. Why is it minus?

Lynn. I don't know. That's just the way we learned it. And see then you're still stuck with the . . . you still have i in the denominator.

Interviewer. And why is that a "+15i"?

Lynn. Because they're both positives . . .

Interviewer. And the last 15i, why is that positive?

Lynn. Because they're still both positive . . .

Interviewer. I don't understand this one then [$-9i^2$]. Why is that one a minus when for everything else you said it's because they're both positive?

Lynn. [After a long pause] . . . I don't know. I don't remember . . . I think we . . . she explained it to us when we were first learning complex numbers . . . she gave us this sort of basic thing to follow and I'm sure she explained how she got to it, but I wasn't paying any attention . . . I wasn't listening and I just figured I knew the formula . . . When she gave us the rule, I had it right then.

## Related Abilities, Attitudes, and Perceptions

Question 6. What evidence of the role of perception and meaning on the solution of algebraic problems is provided by someone who is successful in a basic algebra course.

Question 7. What evidence does someone who is successful in a basic algebra course provide of her attitude towards algebra and her perception of her understanding of algebra?

## Perception and Memory of Algebraic Problems

Lynn was characterized by her rapid perception of the conditions of a task. For example, concerning her rapid solution to Task 1-9 (Find two binomials, $M(X,Y)$ and $N(X,Y)$ such that $M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2$), she remarked:

> It's not like I'm really thinking. It's sort of that . . . it just sort of hits me. Like, I saw this $6X^2 - XY - Y^2$ and these the M and the N multiplied together and I just thought factor. I don't know why - it just sort of came to me. Since I've been doing algebra, all my teachers have always said, "Simplify . . . Try to simplify and try to factor first."

Further evidence of the speed with which Lynn attacked the interview tasks was provided in her solution to the four parts of Task 1-6 (Simplify the following expressions by finding the indicated products and combining similar terms: A) $X^n \cdot X^n$; B) $5X^n(6X^n + 3X^2 + 2X^{n-1})$; C) $(5X^n + 2)(6X^n + 3)$; D) $(X^n + Y^{n-1})^2$). She completed reading the problem and mentally solving all four parts of the task in one minute and 22 seconds. Even her solution to part D was given as if it were being read off of the task sheet with the statement of the problem. After solving the problem she looked back and stared. She noted, "It's just so strange for me to see letters as exponents. Sometimes I have to stop and think about the exponents do I add them or multiply them or subtract them or . . ."

Perhaps one explanation for Lynn's rapid ability to perceive the conditions of a task from the Polynomial Series is her perception of algebraic expressions as variables. In her solution to Task 1-5 (Simplify: $(X + 1)^{1+n}(X + 1)^{1-n}$), she questioned whether X + 1 would be a monomial if it had a 2 as a coefficient. To Lynn the task was asking her to simplify $(A^{n+1})(A^{n-1})$ where $A = X + 1$.

Lynn's "on-the-spot" perceptions of the solutions to tasks was also evident in her solution to the tasks of the Algebraic Fractions Series. However, her desire to have quick flashes of insight often went unanswered due to the complex nature of the tasks. Without the insights she was frustrated and unsure.

Lynn's "on-the-spot" solutions continued in the radical expressions tasks. As if a continuation of reading the problem, Lynn solved several of the radical expressions tasks by compensation. For instance, she solved Task 3-6 (What number multiplied by the product of $\sqrt[3]{4}$ and the $\sqrt[3]{5}$ equals 10?), " $\sqrt[3]{4}$ $\sqrt[3]{5} = \sqrt[3]{20}$ . . . $\sqrt[3]{20}$ times something equals 10 . . . 10x10x10 = 1000 . . . $\sqrt[3]{20}$ $\sqrt[3]{50} = \sqrt[3]{1000} =$ 10 so the solution is $\sqrt[3]{50}$." Similar on-the-spot solutions were recorded on Tasks 3-3 and 3-9.

Not all of Lynn's solutions to the radical expressions tasks were guided by her immediate perception of the solution. In solving Task 3-8 (Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$), she at first stated her solution: "Divided by $\sqrt{2}$. Oh, you could divide as long as the N's are alike $\frac{5\sqrt{2}}{\sqrt{2}} = 4\sqrt{2}$. Well, if you had $\frac{6\sqrt{2}}{2\sqrt{2}}$ they would divide out." She quickly altered her trial solution to $\frac{4\sqrt{4}}{\sqrt{2}}$. She was surprised at the result: "I think my answer would be $4\sqrt{4}$. That would be 8! That doesn't seem right." She went on to check her solution by rationalizing the denominator of $\frac{8}{\sqrt{2}}$.

## Attitudes and Self-Perceptions

Lynn viewed the tasks from the Polynomial Series as easy and fun (Task 1-2, Task 1-8). She tried to solve each task as quickly as possible. Her solutions were at times made so quickly that her statements of the solution sounded as if they were part of the problem she was reading (Task 1-3). Her speech was also rushed and abrupt. It was difficult for her to "think aloud" since her thoughts "flashed by" so quickly even on tasks which were expected to be difficult: "Some of them I'm not thinking anything, 'cause like this one [$5X^n(6X^n + 3X^2 + 2X^{n-1})$]. I mean that I know that I'm multiplying this term by everything in the parentheses, but it's not like I'm thinking about it (Task 1-6)?"

As the tasks increased in difficulty and Lynn began the Algebraic Fractions Series she still had difficulty verbalizing her solutions:

> A lot of times I'll just look at the problem and I can see the numbers in my head and it's just . . . and I'm not really thinking anything . . . there just sort of in there . . . sort of moving around . . . I can see in my head . . . I see it's $\frac{2X + Y}{X}$. I can see it like that. But it's just hard for me to say what it is that I'm thinking, because I'm not really thinking anything. I'm thinking but I'm not consciously thinking of what it is I'm thinking. (Task 2-1)

As the content of the tasks became more difficult for Lynn, she began to have difficulty with the style of presentation of the Algebra Interview Tasks:

> All this when they say 'divided by' this and that and when you divide this into that is what confuses me. I have to stop and think which am I dividing into and which . . . which is the divisor and which is the dividend and all that. If I could figure out what the problem is saying then it's easier for me to do it. (Task 2-3)

It was not until the complex number tasks that Lynn indicated a strong dislike for a specific topic of mathematics. She repeatedly indicated that she "hated" complex numbers (Task 4-1, 4-2, 4-4).

Lynn was not confident in her work in algebra. She was unsure of why she was doing what she was doing and whether it would work. She used a check as a measure of whether her entire process was correct. A "yes" meant the process and solution were correct. A "no" meant that the process was wrong. She never questioned the possibility of mistakes made in performing the steps of her algorithms as a cause of the "no" response (Task 2-16).

Further evidence of Lynn's view of mathematics was provided by her solution to Task 2-10 (Find the algebraic fraction in lowest terms if $\frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21}$ divided by it yields $\frac{2Y - 3}{Y - 7}$). She obtained the incorrect solution $\frac{Y - 3}{2Y - 3}$. She noted, "I don't know what it means . . . I know how to get to the answer, but I don't know why." She checked her answer by multiplying it by $\frac{(2Y - 3)(2Y - 3)}{(Y - 7)(Y - 3)}$: "It works but I don't know if it's right." The interviewer tried to direct Lynn to check her solution for the conditions of the problem. She began to and then stopped and said, "I didn't divide, but it still works out so that's still the answer." Finally she multiplied $\frac{2Y - 3}{Y - 7} \cdot \frac{2Y - 3}{Y - 3}$ to show that her solution worked.

## Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test

Lynn's classifications for the 21 tasks of the Classification Test are presented in Table 4.4. She separated the cards into four piles. She described the four piles, "One of them was problems with complex numbers, another one had problems with radicals in it, another one was simplifying and factoring, and then the other group was something like 'What number times $X^2$ equals X - 5?' or something like that." She used the style of presentation to separate tasks for polynomials and algebraic fractions. Then she assigned all radical and complex numbers tasks to their own pile based solely on the type of algebraic expression.

On the Mathematical Memory Test, Lynn was able to recall much of the syntactic information from the cards that had been circled. In her pile of complex number tasks, she had circled the task "Simplify: (1 + i)(1 + 2i)." In trying to remember the task, she recalled, "The one with the complex numbers was a multiplication problem. One of the terms was 2 + i. I think maybe it was (2 + i)(2 + i). Well

Table 4.4

Lynn's Solutions for the Classification Test and the Preference for Algebraic Problems Test Coded by Type of Algebraic Expression - Polynomial (P), Algebraic Fraction (F), Radical (R), Complex Number (C); Operation - Add (A), Subtract (S), Multiply (M), Divide (D), Factor (F); and Style of Presentation - Textbook (T), One (1), Two (2), or Three (3) Missing Expressions to Find

CLASSIFICATION TEST

PREFERENCE FOR ALGEBRAIC PROBLEMS TEST

| | SIMPLIFY | COMPLEX NUMBER | FIND | RADICALS | |
|---|---|---|---|---|---|
| 1 | P-M-T | | | | 1 |
| 2 | P-F-T | | | | 2 |
| 3 | P-S-T | | | | 3 |
| 4 | | | | R-A-T | 4 |
| 5 | F-D-T | | | | 5 |
| 6 | F-S-T | | | | 6 |
| 7 | | | P-D-2 | | 7 |
| 8 | | | P-M-2 | | 8 |
| 9 | | | P-M-3 | | 9 |
| 10 | | | | R-D-T | 10 |
| 11 | | C-M-T | | | 11 |
| 12 | | C-A-T | | | 12 |
| 13 | | | P-S-1 | | 13 |
| 14 | | | F-D-1 | | 14 |
| 15 | | | F-A-1 | | 15 |
| 16 | | | | R-D-1 | 16 |
| 17 | | | F-S-2 | | 17 |
| 18 | | | | R-M-1 | 18 |
| 19 | | C-S-1 | | | 19 |
| 20 | | | | R-A-1 | 20 |
| 21 | | C-M-3 | | | 21 |

anyway one of them had $2 + i$ in it." When asked if the word "simplify" was written on the card, she responded, "No, I think it just had the problem written there." She was correct in terms of the style of presentation, operation, and type of algebraic expression. Even her choice of complex numbers showed syntactic resemblance to the actual task.

In her pile of radical expression tasks Lynn had circled the task "Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$." While trying to recall the task, she noted that it was not a simplification task. "It was something like, 'Find the number that times the $\sqrt{6}$ equals something minus something.'" She was able to recall the style of presentation, the algebraic expression, and that the $\sqrt{6}$ was involved. She was incorrect in terms of the operation involved in the task.

The circled task in Lynn's simplifying and factoring pile was "Simplify: $\frac{A}{A^2 - 16} - \frac{A + 1}{A^2 - 5A + 4}$." She indicated that she could recall the actual problem. "I think it was $\frac{A}{A^2 - 16} = \frac{A - 1}{A^2 - 4A + 5}$. I think that's what it was . . . something like that. I know that the $\frac{A}{A^2 - 16}$ is right." When asked what the operation was that was involved, she responded, "I cross-multiplied." She was able to recall the problem almost verbatim. Her only errors were in the operation in the numerator of the second fraction and in the main operation of the task. Although none of the tasks involved equations, she recalled the "-" as " = ." She could not recall the circled item in the last of her piles.

After the cards were reshuffled, Lynn arranged the tasks in order from easiest to hardest. She placed those she liked to do best first; for example, "Factoring and stuff like that." The complex number tasks were held until last because she "hates to do complex numbers." Her order of preference is provided in Table 4.4. Her order of preference was simplifying polynomials, simplifying addition of radicals, simplifying fractions, polynomials involving multiplication and division "to find," simplifying division of radicals, simplifying complex numbers, and the remaining algebraic expressions "to find" tasks. The latter group was generally ordered polynomials, algebraic fractions, radicals, and finally complex numbers. She moved two tasks further down, however, because she remembered the difficulty she experienced with them during the interview sessions.

## Summary

Lynn can be characterized by her tedency to use compensation on the Algebra Interview Tasks. On tasks in which she did apply negation she was unsure of the appropriateness of using a different operation to solve a task than the task indicated. In her confusion over the appropriateness of applying negation she frequently reversed her operations. Lynn's reliance on compensation guided her solutions to the tasks designed to measure a student's understanding of the relationship between multiplication and division. She had difficulty applying compensation to multistep algorithms and was unsure of the acceptability of using negation on the multistep tasks.

Lynn demonstrated an ability to generate more than one solution for a problem when requested, but she did not volunteer solutions if one was not specifically requested. She felt very unsure of operations with complex numbers and had difficulty transferring the operations for polynomials to complex numbers.

Lynn viewed the learning of mathematics as learning a system of rules. She perceived tasks syntactically and could remember a great deal of the symbolism used in mathematics, but would leave out significant features of a task.

## Claire

### Reversibility and Flexibility

Question 1. What evidence of a reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

Question 2. What evidence of a flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

### A Pattern of Behavior

In her solution to Task 1-14 (What trinomial subtracted from $5X^2Y - XY^2 + 7$ yields the difference $-X^2Y + 8$?), Claire established a connection between the task and problems from arithmetic. She recalled solving the arithmetic tasks through negation:

> I'm thinking, this used to be in fourth grade on something, we would have problems where you'd have a number, 52, and you'd have an answer, 30 as your answer, and you've got to fill in the blank. And what we do is -- we find out what subtracted from that gives that. So what you do is subtract 30 from 52 and you get 22. No that's not right. Wait a minute. No, it is right. So I ought to be able to do the same thing. I ought to be able to subtract that [$-X^2Y + 8$] from that [$5X^2Y - XY^2 + 7$] and to get the answer.

From this point on Claire tended to apply the negation approach first to interview tasks of the form □ * b = c or a * △ = c, in which a,b,c were given algebraic expressions and * is a given operation, and the student is asked to find the algebraic expression. As shown in Table 4.5, this trend was not true for tasks with two or more unknown algebraic expressions. She generally applied compensation to tasks of the form □ * △ = c and □ * △ = ○, where □, △, and ○ are unknown algebraic expressions, c is a given algebraic expression, and * is a given operation.

### Flexibility of Approach

Claire was persistent in her efforts to solve the algebraic interview tasks. She was reluctant to request hints. When she did ask for some help, she had some difficulty in applying a hint if it required her to switch her approach. She was also persistent in sticking with an approach even if she felt that it was incorrect. For example, in her solution to Task 1-5 (Simplify:

### Table 4.5

### Algebraic Interview Tasks Solved by Claire With Hints (Underlined) or Without Hints (Not Underlined) by Negation (N), Compensation (C), or Another (A) Approach

(The number in parentheses indicates the order of presentation for the task.)

| | | Find the Missing Algebraic Expression: | | | | | Vocabulary and Algebraic Symbolism |
|---|---|---|---|---|---|---|---|
| | | One Expression □ * B = C or A * △ = C | | Two Expressions □ * △ = C | | Three Expressions □ * △ = ○ | |
| | − | 14(11,N) | − | 8(8,C) | − | 2(2,A) | 1(1,A); 3(3,A) |
| Polynomials | × | 12(13,N*); 13(13,C**) | × | 9(9,C); 10(10,A) | × | 4(4,C); 11(12,A) | 5(5,A); 6(6,A) |
| | | | ÷ | 7(7,C) | ÷ | 15(15,N) | 16(16,A) |
| | + | 1(20,N); 5(13,C) | | | | | |
| Algebraic | − | 4(12,C*); 15(22,N) | − | 12(16,C); 13(17,C) | | | |
| Fractions | × | 2(21,N); 14(45,N); 16(14,N*) | | | | | |
| | ÷ | 7(15,N); 10(31,N) | | | | | |
| | + | 4(21,N) | | | | | 1(18,A) |
| Radical | − | 9(34,N) | | | | | 2(19,A) |
| Expressions | × | 3(20,C); 5(22,N*); 6(23,C); | | | | | |
| | | 7(33,N); 11(43,N) | | | | | |
| | ÷ | 8(24,N); 10(42,N) | | | | | |
| | + | 1(25,N) | + | 7(30,C) | | | |
| Complex | − | 4(37,N) | − | 5(28,C) | | | |
| Numbers | × | 2(26,N); 3(27,N*) | | | × | 6(29,C) | |
| | | | | | ÷ | 8(41,N) | |

* First attempted compensation.

** First attempted negation.

$(X + 1)^{1+n} \cdot (X + 1)^{1-n}$) she applied the false generalization $(a + b)^n = a^n + b^n$. Although she believed that something was wrong, she continued with some frustration. In the midst of her attempted solution she commented, "I think I'm really just going way off now. I don't think this is working out right. That's what happens to me on tests. I just get completely off the track and I never realize it."

In her attempted solution to Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?), Claire tried to switch her approach for the task from negation to compensation. This switch was short lived, and she soon came back to her negation approach:

> Now I'm thinking whether I can do this by trial and error [compensation] just by making up something that's completely just drawn out of the blue. Multiplying it by that and trying to get it. I'm trying to say what times 4A is going to give me $64A^3$. That's going to be $16A^2$. But I've got to have a trinomial, not a binomial. Oh, I don't know. I'm rearranging my thoughts. I'm trying to go at a different angle now and I can't think what angle to go at. I keep wondering why this doesn't work. Maybe I'm doing my division wrong [back to negation]. This is the right idea. I've got the right idea of dividing but I can't make it work for me. I don't like to let these things beat me.

On the algebraic fractions tasks, Claire was persistent in her attempts to solve the tasks by compensation, but she was not as persistent with her use of negation. In her solution to Task 2-4 (Find the algebraic fraction in lowest terms if $\frac{2X + 3}{X^2 - 3X + 2}$ subtracted from it yields $\frac{1}{X^2 - 3X + 2}$), she first attempted to solve the task by negation and then switched to compensation:

> If I had $x - 2 = 1$, all I do is add the 2 and the 1 to get that so it seems like I ought to be able to add that [$2x + 3$] and that [1]. But that won't work out right I don't think. To get that out of there. Let me think now, I'm trying to picture something that will end up being 1 when I subtract that [$2x + 3$] from it [writes: $\quad - 2x + 3 = 1$]. I need to get rid of the 2x. So I could put 2x over there. So $2x - 2x$ would be 0 and that get's rid of that. And I could say what minus 3 is 1, so that would be 4."

The persistence that characterized Claire's behavior on the polynomial and algebraic fractions tasks was not as evident in her attempted solutions to the radical expressions and complex number tasks. Her sporadic difficulty with transferring the rules of polynomials to her work with radicals and complex numbers left her apparently relying on specific hints to direct her solutions. For example, on Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals -14?), she initially tried a form of compensation to find the number. After two attempted false transfers, she switched her approach to negation. Although she knew that she could get her answer by dividing -14 by $2 + 3\sqrt{2}$, she lamented, "How in the world you do that, I do not know." The interviewer attempted to provide general hints to help Claire solve the problem, but she jumped from one approach to another. In one case, she set up the equation $(2 + 3\sqrt{2})x = -14$, but she did not attempt to solve it. Guided by very specific hints by the interviewer (for example, "Try multiplying by $2 - 3\sqrt{2}$."), Claire obtained the correct answer. Afterwards, she reacted bewilderedly, "And that's that, but where . . .? But I don't know how I got that. That was just 'happen chance' or whatever. I didn't do that. I mean -- that was definitely a surprise answer."

## Reversibility of Multiplication and Factoring

Claire viewed the relationship between the reversible operations of multiplication and factoring as related, but she did not completely understand the implications of that relationship. On Task 1-11 (Write a binomial P(X,Y) and a binomial Q(X,Y). What is the product of P(X,Y) and Q(X,Y)? Next, take the product you've just found and factor it) She selected the binomials $3X + 2Y$ and $4X + 5Y$. Next, she applied the FOIL method to multiply her binomials. She wrote, "$12X^2 + 15XY + 8XY + 10Y^2$." After combining similar terms, she began to factor her trinomial. First, she searched for common factors. Then, after a pause of several seconds, she announced that she knew that the factors would be her two binomials. She then went on to factor the trinomial as if she had never seen the two binomials. She had some difficulty in factoring because she couldn't come up with a middle term of 23XY. The problem took her about 5 minutes to solve.

## Reversibilty of Multistep Algorithms

In her work with algebraic fractions Claire experienced a great deal of difficulty in applying compensation to the tasks. Her difficulty was largely due to her inability to reverse her thinking process on multistep tasks. She would write the initial statement of a problem and attempt to solve the problem as if there were no intermediate steps. For example, in her solution to Task 2-5 (Find the algebraic fraction in lowest terms that when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$) she persistently attempted to solve the task by compensation. First, she wrote the indicated addition as:

$$\underline{\qquad\qquad} + \frac{X(X-1)}{(X-5)(X-1)} = \frac{2X - 15}{X^2 - 6X + 5}.$$

Next she decided that the denominator of the missing fraction was X - 1. This assumption blocked her in resolving the problem. She was unable to handle the conditions that the sum was to be $\frac{2X - 15}{X^2 - 6X + 5}$ and that she already had $\frac{2X^2 - X}{(X - 5)(X - 1)}$. She could not coordinate the conditions, and instead viewed them as contradictory. Despite the fact that the use of negation was a much easier approach to Task 2-5, she persisted in her attempt to solve the task by compensation. Her eventual solution of the task required a great deal of interviewer direction. After solving it by compensation, she still felt uneasy about her solution and suggested checking it by negation. She quickly verified her solution by negation.

Claire's solution to Task 2-6 (Find the algebraic fraction in lowest terms which multiplied by $\frac{X - 2}{X^2}$ yields $\frac{X - 1}{X^2 - X}$) provided further evidence of her difficulty in reversing multistep algorithms. Once again she was unable to coordinate the conditions of the problem:

> What do you multiply by $X^2$ to get $X^2 + X$? I don't know. I also don't know what times $X - 2$ gives you $X - 1$. The thing that bothers me about this is that I understand this. I understand that multiplication is easier than addition for one thing 'cause it's simpler. There's no common denominator to be found. What I can't see is how you can multiply that and get $X - 1$. You can't multiply anything by 2 and get 1. Now wait a minute . . . ½.

The interviewer attempted to help Claire realize that an intermediate step might connect the statement of her indicated multiplication and the given product. Even with hints, she did not acknowledge that she might simplify her multiplication to obtain $\frac{X - 1}{X^2 + X}$. In the midst of her effort to solve the task with hints, she suddenly switched approaches and solved it by negation. It was only after she had obtained the fraction $\frac{X(X - 1)}{X - 2}$ that she was able to follow what happened during the multiplication process.

Another example of Claire's difficulty in coordinating the conditions of a problem was provided by her solution to Task 4-6 (Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$, such that their product is a negative integer). She solved Task 4-6 by compensation. Before trying any pairs of complex numbers she verbally established a set of conditions for the task: i's must disappear, everything else must combine to be a negative integer, the coefficients won't make any difference, the sign of the $i^2$ term should be positive, and one of the i terms must be positive and the other must be negative. She did not overtly note that her last two conditions were contradictory. After stating the conditions of the problem, she began to try various possibilities until she hit a pair that worked. "Since the coefficients didn't matter," she held them constant in all her trials. Once she began generating possibilities she ignored the other "established" conditions. Her list consisted of: (4 + 3i)(4 - 3i), (4 + 3i)(4 + 3i), (-4 + 3i)(4 - 3i), (4 - 3i)(4 - 3i), (-4 + 3i)(-4 - 3i), (-4 + 3i)(-4 + 3i), and finally, (-4 + 3i)(4 + 3i).

## Ability to Generalize and Transfer

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

Question 5. What evidence of the use of false generalizations and false transfer is provided by someone who is successful in a basic algebra course?

### Ability to Deduce the General from the Specific

When coaxed by the interviewer, Claire generalized her solutions to other specific cases but tended not to make general solution statements. For example, on Task 1-2 (Write a binomial in X and Y where one of the terms is a constant. Subtract from your first binomial another binomial in X and Y where one of the terms is a constant. Simplify your expression by combining similar terms), she extended her solution to terms with larger numerical coefficients and to negative constants, but did not state her solution in general terms.

On Task 1-1 (Write a polynomial of degree 6 which is degree 2 in X, degree 5 in Y, and degree 4 in Z), Claire first wrote $X^2Z^4 + Y^5$. After being asked if her answer was unique, she found that she could also solve the problem by introducing new variables or other powers of the variables ($X^2W^2A^2 + Y^5 + Z^4$ and $X^1Y^5 + X^2Y^3 + Z^4$). However, she did not state a general solution to the task.

There were two instances in which Claire did state her solution in general terms. She solved Task 1-7 (Find two monomials whose quotient is $-5AB^2C$) in general terms: "Anything you want to put, when you divide it you get five and make sure whatever your exponents are on the variable that when you subtract them you get $A^1B^2C^1$. It's just a method of trial and error [compensation]." Then she was asked to write an answer using "n" as an exponent. She demonstrated that she could use variables as a way to generalize her solution to a specific task.

Claire also provided a general solution for Task 4-5 (Find two complex numbers of the form A + Bi where A ≠ 0 and B ≠ 0 such that their difference is -2i). In this case, she volunteered that her first solution was not the only solution. After giving several other alternatives she stated, "You could make up anything -- just so you came up with 0 for the A number and -2 for that. And there's a lot of solutions for that."

### Ability to Generalize from Task to Task

Claire was not consistent in her generalization of the processes that she used to solve the Algebra Interview Tasks from task to task. She experienced a great deal of difficulty in her attempt to use compensation to solve Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$). After obtaining her solution, she felt uncomfortable with it: "I keep thinking you can't add a trinomial and a binomial. Or maybe you can? I guess that's right. I don't know if I did it right." She quickly checked her solution by negation: "I should be able to subtract the second term from the last one and get this first one that I made up."

Despite the ease with which she applied negation to check her solution to Task 2-5, Claire did not generalize it's application to Task 2-6 (Find the algebraic fraction in lowest terms which multiplied by $\frac{X - 2}{X^2}$ yields $\frac{X - 1}{X^2 + X}$). Instead, she unsuccessfully attempted to apply compensation to solve the task. In the midst of following a chain of hints provided by the interviewer, Claire exclaimed, "Wait! Whoa! When one multiplies, just say for instance we have '? x 3 = 12,' what one does is divides 3 into 12 to get that answer. So I can do the same thing." She then solved the task by negation. In this case, rather than generalize her process of solution from one task in the algebraic fractions domain to another, she transferred her solution process from processes used to solve arithmetic tasks.

At the beginning of the next interview session, Claire generalized her method of solution to Task 2-7 (Find the algebraic fraction in lowest terms if $\frac{X^2 - XY}{XY}$ divided by it yields $\frac{X^2Y}{3}$ ): "I'm going to see if you can do the same thing that I did yesterday and just multiply instead of divide to get the answer. Alright, 12 divided by something equals 4. Then you divide 4 into 12 to get it. If I could divide that [$\frac{X^2Y}{3}$] into that [$\frac{X^2 - XY}{XY}$] . . . which is the same thing as multiplying this by the inverse of that."

Ability to Transfer

Claire would frequently use arithmetic tasks to provide her some directions for her solution to algebraic tasks. The process was not always successfully transferred. On Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$), she tried to solve a simpler problem involving numbers:

> Okay my goal is say 7/8 and then I want something with a different denominator. I could really make it hard and put something like . . . no I can't do that because it's got end up being 8. I started to say I could put a 3 or something. That way I'd have to multiply and I'd get 24 for the common denominator and that's not what I want. However, that could reduce back down to 8 maybe. I'll make it simple and just put 4 for the denominator here

> and 3/4 . . . a different denominator but it's still easily divisible into 8. And then I want something that added to that gives me 7/8. So it'd be 1/8 [writes 3/4 as 6/8]. But with all this other junk [refering to the variables and coefficients] I don't see how you can get this if the denominator isn't X - 1 because that makes up the common denominator of $X^2 - 6X + 5$. Well I could say that I could put this [$X^2 - 6X + 5$] as the denominator. No, I couldn't 'cause when you add it just doesn't work that way.

Claire never recognized that this was analogous to her solution to her numerical example. Although she was able to solve her numerical variant of Task 2-5 by compensation, she was not able to transfer her solution to the algebraic variant.

Claire's reliance on solving related arithmetic tasks as a heuristic for solving the algebraic tasks was also evident in her solutions to the tasks in the radical expressions and complex number series. In trying to solve Task 3-2 (For a radical expression to be in simplest form it must be the case that a radical does not appear in the denominator of a fraction. Find a radical expression equal to $7\sqrt{2}$ which is not in simplest form since a radical appears in the denominator of a fraction), she was unable to think of any algebraic expressions that would need to be simplified because of having a radical in the denominator. To solve the task she thought of other ways to represent the indicated multiplication of $7\sqrt{2}$. This led her to the general statement $a \cdot b = a \div \frac{1}{b}$; where, for Claire, a and b represented natural numbers. After a long pause, she said, "What I want to say is . . . (another long pause) . . . . I know when you have . . . . you do something . . . you turn something over and you multiply when you have these fractions and you'll come out with the same thing. I believe that's right. That'll be the same thing as $7 \cdot \frac{\sqrt{2}}{1}$ which is that [$7\sqrt{2}$]." She transferred her processes and operations for numerical examples with "complex" fractions to her work with radicals.

Claire transferred many of her operations and procedures for polynomial expression to her solutions on the radical expressions and complex number tasks. She explained her view of the relationship between the addition of complex numbers and polynomials in her solution to Task 4-7 (Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ such that the sum of the complex numbers is i):

> I knew what would work. This kind of addition is easier. Sometimes addition is more difficult, but this was easier. If all you've got to do is get a one, then you know just to make these other terms disappear and have this be the difference of 1. And i doesn't have any bearing on it. 'Cause you're not fooling with it. Another time I was fooling with the $\sqrt{-1}$ or $i^2$ which is -1 or whatever. But since i didn't have much to do with it, except as the answer, it was easy.

Further evidence of Claire's ability to transfer operations and procedures is provided by her solution to Task 4-3 (Find the complex number which multiplied by $5 + 3i$ equals $13 + i$). After many false starts, she eventually obtained $\frac{13 + i}{5 + 3i}$. After attempting to rationalize the denominator by multiplying by $\sqrt{-1}$, she multiplied by $5 - 3i$. When asked where the $5 - 3i$ came from, she explained:

"'Cause I know that multiplying binomials, whatever you have, if its positive over here, you just make it negative and it's going to make the middle term drop out. And I kind of just had a hunch that it would work because they're handy little things to have around." After multiplying she obtained $\frac{68 - 34i}{34}$ and noted that she could not reduce by just cancelling the 34's. In addition to transferring many correct operations and processes from polynomials to radical expressions and complex numbers she transferred many incorrect processes and operations. These "false transfers" and "false generalizations" will be discussed in the next section.

## Errors, False Generalizations, and False Transfers

Claire made the following false generalizations: combining unlike terms (Task 1-10), raising each term of a binomial to the power of the binomial (Task 1-5), changing the signs of the terms of a binomial being added to another after dropping the parenthesis (Task 1-6), "cancelling" in an algebraic fraction (Task 1-12), applying the FOIL method to the multiplication of monomials (Task 2-7), and squaring a cube root to rationalize a denominator (Task 3-6). In the first three cases the false generalizations occurred on tasks in which she was experiencing difficulty with the use of variable exponents in the tasks. In the next case she viewed the task as division instead of reducing a fraction.

In the fourth instance her attention was dominated by the complexity of working with algebraic fractions. And in the last instance she was perplexed by the process of rationalizing a denominator.

In Claire's solution to Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?), she was focusing on the algebraic expression $\frac{64A^3 + Y^3}{4A + Y}$ as an indicated division and did not think of it as a fraction. Hence, when she attempted to divide she did not apply a rule she had previously demonstrated that she knew for reducing fractions. "Now I know that when you're doing this you have to be careful not to factor when you're adding but I think that I can factor the 4A out of this side and the Y out of this side."

In her solution to Task 2-7 (Find the algebraic fraction in lowest terms if $\frac{X^2 - XY}{XY}$ divided by it yields $\frac{X^2Y}{3}$), Claire's attention was centered on making sure that she applied negation properly to the task. In multiplying the denominators of the fractions she incorrectly applied the "FOIL" method to find the product of XY and $X^2Y$. Despite the fact that she spent several seconds looking back over her work, she never noticed that the denominator $X^3 + XY + X^2Y + Y^2$ was in error. In this situation, a procedure she learned after she learned how to multiply monomials later became a "bug" in her procedure for multiplying monomials.

Claire made two types of false transfers in her solutions to the radical expressions and complex number tasks. The first was to transfer processes appropriate to the solution of equations inappropriately to operations with algebraic expressions. The only evidence of her application of this form of false transfer was found in her solution to Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals $-14$?). She was trying to find the number by first finding a way to get "rid of the radical" in the problem. She attempted to apply a heuristic that she had recently learned in her basic algebra course -- try squaring both sides. In class this technique had been applied to "both sides" of an equation to "rid" the equation of radicals. She applied the rules to Task 3-4 by squaring "both sides" of $2 + 3\sqrt{2}$; $(2)^2 + (3\sqrt{2})^2$.

The second type of false transfer made by Claire in her solutions to the radical expressions and complex number tasks was to transfer incorrect processes apparently made while learning to solve polynomial tasks to the radical expressions and complex number tasks.

The application of these false processes to polynomial tasks may have appeared to be extinct, but the memories of these false operations and processes were apparently still available for transfer to new domains. A listing of some of the false transfers made by Claire on the radical expressions and complex number tasks is provided in Table 4.6. The polynomial equivalent, actual error made, and the task on which it was made have been provided in the Table.

Table 4.6

Claire's False Transfers of Processes for Polynomials to Processes for Radical Expressions and Complex Numbers on the Algebra Interview Tasks

| Process with Polynomials | Error Made | Algebra Interview Task |
|---|---|---|
| $A + BX = (A + B)X$ | $2 + 3\sqrt{a} = 5\sqrt{a}$ | Task 3-5 |
| $BX + A = (A + B)X$ | $117i + 25 = 142i$ | Task 4-3 |
| $(A + BX)X = A + BX^2$ | $\frac{13 + \sqrt{-1}}{5 + 3\sqrt{-1}} \cdot \frac{\sqrt{-1}}{\sqrt{-1}} = \frac{13 - 1}{5 + 3}$ | Task 4-3 |
| $(A + B)(C + D) = AC + BD$ | $(13/5 + 1/3)(5 + 3i) = 13 + i$ | Task 4-3 |
| $A + BX^a = A - BX^a$, if $X < 0$ | $-3 + 4 \cdot -1^{\frac{1}{2}} = -3 - 4^{\frac{1}{2}}$ | Task 4-1 |
| $(X^a)(Y^b) = (XY)^{ab}$ | $20^{\frac{1}{3}} \times \frac{1}{2}^{3} = 10^{1}$ | Task 3-6 |
| $\frac{X^a}{Y^b} = \frac{X}{Y}^{a-b}$ | $\frac{10^{1}}{20^{3}} = \frac{1}{2}^{\frac{2}{3}}$ | Task 3-6 |

Claire worked for 15 minutes on Task 3-6 (What number multiplied by the product of $\sqrt[3]{4}$ and $\sqrt[3]{5}$ equals 10?). The following excerpt from the transcript of her solution to Task 3-6 is included here as evidence of the frustration she experienced in trying to transfer the processes she used in her work with polynomials to radical expressions:

> Claire. I want to multiply it [$20^{\frac{1}{3}}$] by something so that the exponent disappears or equals 1. That would be . . . . . ...
>
> Interviewer. Could you explain again how you multiply these two together?
>
> Claire. Multiply the base times base, the exponent times the exponent.
>
> Interviewer. Is that a rule . . .?
>
> Claire. I mean, no, multiply the base times the . . . Whoa, okay now I see what you're saying. You've got to add exponents . . . Alright . . .
>
> Interviewer. Well, you add exponents if the bases are the same. Like if you have $(x^2)(x^3)$, then it's $x^5$.
>
> Claire. Can't you just multiply the . . . Multiplication is supposed to be so easy!!! Can't you just multiply them together and just add the exponents.
>
> Interviewer. Not with different bases though.
>
> Claire. Ahhh! I can see why you say that in addition. Multiplication is just a simple way to do things. I can go back to the old $10/\sqrt[3]{20}$. That's going to give me an odd answer, whatever it is. Which is the same thing as $10^1/\sqrt[3]{20}$ . . . Which I could say $10^1/20^{\frac{1}{3}}$ and that's going to give me the same thing.
>
> Interviewer. The same thing because you're using the same rule - but that isn't . . ..
>
> Claire. . . .that don't work?
>
> Interviewer. . . . No, it's not a rule.

Claire. Arghh!!! I'm making them up as I go along. That just looks like it would work!!

Interviewer: Well I'll tell you what -- try a numerical example, like, take $2^3 \times 3^2$. In other words, where you have different bases you have 2 in one base and 3 in the other base. Now see if the rule that you're trying to use works with that case; because there you can tell what the numbers are.

Claire. Not if these . . . had to be that . . . because that's what this is. Have to be $(2^3)(\frac{1}{3}^2)$, which is $2^3/3^2$. Okay. Well, alright, when you divide you subtract exponents.

Interviewer. But only if you have the same base.

Claire. But that . . . it wouldn't make any difference, if you had the same bases, then it'd be 1.

## Related Abilities, Attitudes, and Perceptions

Question 6. What evidence of the role of perception and memory on the solution of algebraic problems is provided by someone who is successful in a basic algebra course?

Question 7. What evidence does someone who is successful in a basic algebra course provide of her attitude towards algebra and her perception of her understanding of algebra?

### Perception and Memory

Claire's initial perception of a task was a strong influence on her continual perception of a problem. For example her initial perception of Task 1-9 (Find two binomials, $M(X,Y)$ and $N(X,Y)$, such that $M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2$) was that the problem was asking her to do something she had not been required to do in the past. The problem was requiring her to "make up" her own multiplication problem. After initially reading the task, she commented, "They're real familiar, but making up one of my own is the hard part. I've never had to go backward like this. I'm going to start experimenting and just put something down." After Claire solved the task, the interviewer asked Claire if she had ever solved any problems like this before. She responded, "I've done a lot of multiplying binomials together. But I've never had to start and make up one myself. It's kind of like going backwards I guess." She apparently linked the task with the multiplication of binomials and not with the factoring of polynomials.

Another example of the influence of Claire's initial perception of a task on her solution to a problem was provided by her attempted solution to Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?). She initially viewed the problem as one that could be solved by dividing $64A^3 + Y^3$ by $4A + Y$. She then wrote the indicated division in the form of a fraction. Once she viewed the division as a fraction she was unable to break this fixation even though she could not factor the numerator of the fraction. Each attempt led to the binomial $16A^2 + Y^2$ as the solution and since the statement of the problem directed her to find a trinomial she knew that this answer would not meet the condition of the problem. It was not until the interviewer suggested long division that Claire was able to break her fixation and solve the task. She lamented, "I don't know why I didn't think of this. We've done this long division in class."

Claire was unable to solve Task 2-5 (Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$) because she was unable to break from her initial perception that if X - 5 is one of the denominators, the missing fraction must have a denominator of X - 1.

Claire was unsure of the use of i in an indicated division. Although she immediately applied negation to the conditions of Task 4-2 (Find the complex number which when multiplied by 3 - i equals 10), she was unsure of what to do with the resulting algebraic expression $\frac{10}{3 - i}$. As soon as the interviewer questioned what "i" meant, Claire completed the task. She rewrote the algebraic expression as $\frac{10}{3 - \sqrt{-1}}$, setting in motion an algorithm for solving the task. First, she multiplied numerator and denominator of the fraction by $3 - \sqrt{-1}$. When that didn't rationalize the denominator she multiplied the numerator and denominator of the fraction by $3 + \sqrt{-1}$. After she multiplied in the denominator her voice began to get excited: "Hey now wait a minute! Do I see something here? This 10 will disappear and you end up with $3 + \sqrt{-1}$ or 3 + i. Hey! So (3 + i)(3 - i) = 10! You never could have told me that! I never would have believed it! That does not look like it equals 10!"

Claire was frequently bothered by a feeling that her problems were not complete (a feeling of "lack of closure").

In working with polynomial expressions, she expressed a feeling that polynomial expressions seemed incomplete if they can be factored. For example, while solving Task 1-3 (Write a trinomial which contains an $X^2$ in each term and yet contains no like terms), she answered:

> $5X^2 + 3X^2Y + X^2Z$. But I'm not at all sure that's right because you could factor this out. You could factor the X 's out. And I thought you couldn't do that if the terms . . . No, that's addition where you can't add unlike terms. See I'm getting my multiplication and my addition mixed up now. So I guess that's right. I don't know what else to do . . . I feel like I want to factor them out. And you get $5 + 3Y + Z$ [student writes $X(5 + 3Y + Z)$], which doesn't mean anything. But just to play with it, that's what you get. I guess what I don't understand is what like terms are in trinomials. I know you can't add unlike terms and that's what you're essentially trying to do, but when you factor, that's multiplication.

In her solution to Task 1-6, part b (Simplify the following expressions by finding the indicated products and combining similar terms: $5X^n(6X^n + 3X^2 + 2X^{n-1})$), Claire again noted her feeling that polynomials written with common factors in each term looked incomplete unless written in a factored form. Despite the fact that she multiplied correctly she felt as if her solution wasn't finished. "That doesn't look right. I get these things mixed up. My tendency is to factor everything out that you can, but you can't do that with addition. When you factor you do that with multiplication. I had to remember that so many times."

Task 3-4 (Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$) was designed to create in the student a feeling of "lack of closure." Of interest was the type of errors that a student would be willing to make in order to achieve closure. Claire applied several false generalizations in her attempted resolution of the lack of closure which the task created for her: "It seems like you'd just add $\sqrt{3} + \sqrt{3} = \sqrt{6}$." After the interviewer corrected the error, Claire continued, "Well the $\sqrt{3} + \sqrt{3}$ would that be 3? No, that's multiplying. $\sqrt{6} - \sqrt{3}$ is the number we're looking for . . . Can I put the 6 and 3 under one thing? Like if this were multiplication I could put $\sqrt{6 \cdot 3}$ but that doesn't work with this though does it?" Again the interviewer corrected her. Although the interviewer questioned whether Claire had ever left answers in a form like $\sqrt{6} - \sqrt{3}$ before and whether the $\sqrt{6} - \sqrt{3}$ met the conditions given in the problem, she felt as if she had not yet completed the problem: "I didn't get the answer though. What I have would fit, but I just don't have any answer . . . . The problem with these things is that they don't have any value. They just don't mean anything so I don't know what to do with them . . . . I'm not satisfied with that, but we can go on if you want to.

### Attitudes and Self-Perceptions

Claire viewed the learning of algebra as learning to follow rules:

> Taken out of context like this, kind of out of practice, it's hard for me to remember the rules. But when I'm in class all I do is just like a little machine. I just follow the rules. And sometimes I don't understand and sometimes I do.

> Just as long as I follow the rules and get it right that's all that I've got to do. (Task 1-8)

> I know the rules. I just don't know why I put down what I do sometimes. I just follow the rules blindly. (Task 1-2)

> This is part of the problem of drawing these things out of context and just putting them on little sheets of paper. It's hard for me to remember what rules go with what things. In class, in tests, and in the book, it's a lot easier. For one thing, you can always look back and find the answer or whatever. And when you just pull them out of context it's hard to remember. (Task 3-4)

Claire was not confident in her mathematical ability. Although she did well on tests she was uncomfortable with her own reliance on meaningless rules. She noted:

> So now you see what I mean when I say that I don't trust my math because I automatically do something, but I'm not sure if it's right. I don't know that what I'm doing is correct or the way to do things. (Task 1-3)

> I think the thing that bothers me is the fact that I have to manipulate things myself and I'm use to following rules and having the numbers do things by themselves and I'm just pulling strings, sort-of. And I'm controlling things more now. I usually just follow the rules and now I make up my own rules almost. In doing regular problems if you do certain certain things, certain things happen. It's almost like it's out of your control. This is all in my control. But I don't know what to do. I don't feel confident in my position. (Task 2-5)

In her solution to Task 3-3 (What number multiplied by $4\sqrt{2}$ equals $24\sqrt{6}$?), Claire noted the subconscious manner in which she applied the rules for multiplying radicals. The following dialogue is an excerpt from the transcript of her solution to Task 3-3:

> Claire. My first inclination is to say, "What times 4 equals 24?" and that's 6. And "What times $\sqrt{2}$ equals $\sqrt{6}$?" and it looks like it'd be $\sqrt{3}$. But I don't know if that's how you do those or not.
>
> Interviewer. Do you remember the rules for multiplying . . .?
>
> Claire. I don't remember rules. I remember them just long enough to take the test, that's my problem. That's probably the problem with a lot of people. I can remember how to add and subtract regular numbers, and that's about it.
>
> Interviewer. Okay, well in fact, that is the rule for multiplying radical expressions. You multiply the coefficients . . .
>
> Claire. Just multiply like times like, right? Well I must remember it then . . . Maybe it's just sort of in my subconscious now and I don't have -- I can't just spout it out -- I just do it.

Three weeks preceding this interview Claire had received a 93 out of 100 on a test in her basic algebra course. The test included problems on multiplying radical expressions. A copy of the test is included in Appendix E.

Claire indicated that part of her difficulty with learning algebra was that she lacked an overview for the tasks of algebra:

> You have to have a big picture. You have to have a broad understanding of the problem. I wade thru problems. I can't see the forest for the trees. I need that broad perspective . . . Teachers should teach from the aspect of letting the student understand what's going on and not just following rules and pushing thru the problem. But giving a broader understanding. And that's so hard to do . . . It's taxing. It's just too hard to understand. (Task 2-6)
>
> I don't have any kind of sense of how to find things, that omniscient viewpoint of math. I'm way down here looking at it. I can see all this stuff. I know it's there somewhere. I know it's

> possible to find things to talk about. But I just don't know how to get there. (Task 2-12)

Claire identified several specific topics of algebra in which she felt confused: subtracting negative numbers (Task 1-2), using variables as exponents (Task 1-6, Task 1-10), and radical expressions (Task 3-3, Task 3-4, Task 3-5, Task 3-6, Task 4-2). She also indicated that she "loved to cancel" on problems involving multiplication of algebraic fractions.

In solving task 1-10 (Factor the following binomial into a product of a monomial and a binomial: $A^{m+3} + A^3$), Claire provided an indication of the abstract quality that she attaches to variables:

> You've learned something interesting about the way people are afraid or don't want to work with things that don't have any value. 'Cause see if it's numbers I can do it. If it's all variables like that I have a hard time with it . . . It was really so simple. I have a mental block about it.

Claire's feelings about working with radicals permeated her solution to the entire series of tasks on radical expressions. Evidence of her feelings about radicals ranged from tonal qualities as she initially read each task to direct statements made in the midst of her solutions: "I hate these things."

Claire also expressed some concern with her ability to solve tasks in the style presented in the study:

> All this working things backwards is just a baffle to me. If you'd ask me to add it up I could do it. But getting what you add up to get it, I can't do that. (Task 2-5)

> I hate these things where you have to go backwards, where I have to make it up. I could do it if they had it over here and they said simplify. I could do that. I need one that's not simplified so I can look at it and apply it. (Task 3-2)

### Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test

Claire's classifications for the 21 tasks of the Classification Test are presented in Table 4.7. She separated the cards into seven piles. The first two piles were categorized by Claire as containing the tasks involving simplification of algebraic expressions. The first pile included addition and subtraction tasks and the second pile contained multiplication and division tasks. Addition and subtraction were put together because as she stated, "that's basically the same function or the same thing." Multiplication and division were put together for the same reason. As a clarification, she noted that in simplifying fractions with addition and subtraction you need to find common denominators.

The next four piles were similar in that in each case "the answer was given and one of the other two terms was missing." Each pile contained tasks for one of the basic operations of addition, subtraction, multiplication, and division. These four piles were separated by the individual operations because "it would simplify things." The separation was based on how "you do different things to find the answer." As she put it:

Table 4.7

Claire's Solutions for the Classification Test and the Preference for Algebraic Problems Test Coded by
Type of Algebraic Expression - Polynomial (P), Algebraic Fraction (F), Radical (R), Complex Number (C);
Operation - Add (A), Subtract (S), Multiply (M), Divide (D), Factor (F); and
Style of Presentation - Textbook (T), One (1), Two (2), or Three (3) Missing Expressions to Find

**CLASSIFICATION TEST**

| PREFERENCE FOR ALGEBRAIC PROBLEMS TEST | SIMPLIFY + or - | SIMPLIFY x or ÷ | ANSWER GIVEN: ÷ | ANSWER GIVEN: - | ANSWER GIVEN: x | ANSWER GIVEN: + | FACTOR | |
|---|---|---|---|---|---|---|---|---|
| 1 | | P-M-T | | | | | | 1 |
| 2 | | | | | P-M-3 | | | 2 |
| 3 | | | | | | | P-F-T | 3 |
| 4 | | | | | P-M-2 | | | 4 |
| 5 | P-S-T | | | | | | | 5 |
| 6 | C-A-T | | | | | | | 6 |
| 7 | | | | P-S-1 | | | | 7 |
| 8 | | | | C-S-1 | | | | 8 |
| 9 | | F-D-T | | | | | | 9 |
| 10 | | | F-D-1 | | | | | 10 |
| 11 | | | P-D-2 | | | | | 11 |
| 12 | | | | | | F-A-1 | | 12 |
| 13 | F-S-T | | | | | | | 13 |
| 14 | | | | F-S-2 | | | | 14 |
| 15 | R-A-T | | | | | | | 15 |
| 16 | | R-D-T | | | | | | 16 |
| 17 | | | | | | R-A-1 | | 17 |
| 18 | | | | | R-M-1 | | | 18 |
| 19 | | | R-D-1 | | | | | 19 |
| 20 | | C-M-T | | | | | | 20 |
| 21 | | | | | C-M-3 | | | 21 |

> If you had what plus four equals seven, you would subtract four from seven to get the answer. And if it was what minus ten equals some number, then you would add to get that number. And the same thing is true for multiplication. And it's not different for simplification. It's just about the same thing. Even though addition and subtraction are different from each other, you've got so many negative and positive numbers that it's all the same.

The last pile contained a factoring problem. She noted that she could have put the factoring problem in with the simplification of multiplication and division because "essentially you're just dividing." Perhaps because of the difference in style of presentation, she did not suggest placing the factoring item with the other tasks involving factoring in the "answer given-multiplication" tasks.

In discussing her solution to the Classification Test, she noted: "I could go back and rearrange them all and do all the ones with fractions together. But that would be a lot harder to learn doing it that way. You learn your operations first." After the experimenter pointed out that this was not the way the textbook was ordered, she responded with some surprise:

> Maybe that's why I'm having so hard a time. In organizing my thoughts I organize the rules for addition together and the rules for addition apply, like the rules of multiplication apply, to just about whatever you're doing. And maybe it's better to go ahead and learn everything there is to know about fractions first, as your going along. But when you end up, you end up using the rules for addition wherever you are or the rules for division wherever you are for whatever you're doing.

Claire's solution to the Mathematical Memory Test was limited by the manner in which she completed the Classification Test. She explained, "During the first couple of piles I purposely picked up a card knowing that I could find cards that looked like that. But as I went on, what I had done really was grouped them any way and it didn't really make any difference which one was circled because they were the same. So I just circled any one of them." She circled a 2 on an arbitrary card but had already decided how to classify other cards like the circled one so she did not realize what it said. She was only able to recall some of the details of the circled card from the first pile.

> I think it was two fractions. It was either addition or subtraction. It might have been subtraction. I know it was fractions. They were algebraic, because they had variables. I didn't pay that much attention to them. I think they had more than one variable, too. It seemed like I saw more than one variable. I was looking for very, very broad classifications. They were at least binomials in the numerators and denominators. I know one of the numerators was a trinomial and then all the other numerators and denominators were at least binomials. I think it was something X. It would have taken a good deal of factoring or something before you simplified it.

Claire's description was initially correct. She remembered the content domain and the operation for the task. Beyond that she remembered the general appearance of the task and that it would require a number of steps to simplify it. She did not remember the particular details of the fractions such as the variable, the number of variables, nor any of the numerators or denominators.

After the cards were reshuffled, Claire arranged the tasks in order from easiest to hardest. Her order of preference is presented in Table 4.7. She noted that the easiest "surprisingly enough" was multiplication of binomials. She put radicals and "working with i" last because, "that's probably the most difficult to understand." Unlike her categories for the Classification Test, the operations did not play a major role in her order of preference.

Claire indicated that it was difficult for her to decide whether to put the textbook task of multiplying binomials or the textbook factoring task first because they were "so much alike." In situations where she considered the task to be of equal difficulty she ordered them in terms of how she would teach someone else algebra.

In looking back through her ordering Claire was surprised to see that she had ordered an addition of complex numbers task and a subtraction of complex numbers task, sixth and eighth respectively. She explained that she had not realized that these were complex numbers tasks when she first went through and treated them as if they were polynomials. "If that were some other number there it would be okay, really we're leaving 'i' to last because 'i' is a little harder to understand."

Claire's intended order was multiplication and division of polynomials, addition and subtraction of polynomials, multiplication and division of fractions, addition and subtraction of fractions, radicals, and finally, complex numbers. She noted:

> Radicals are a little harder to understand so they're all together. A radical is a radical and they're hard to understand sometimes. But after you've gone through all this other division, addition, multiplication, and subtraction you've built up the skills that you need to work with those operations. So what you need to do now is to learn what radicals are. If you learn what radicals are, you already know how to add, subtract, multiply, and divide, so . . .

The style of presentation did not enter into Claire's order of preference. She did not perceive that the words made a problem more difficult. She explained: "It might go back to the old fear of word problems. I have overcome that just recently in class. I finally decided that word problems are not that much different from anything else. Way, way back -- like in fourth grade when you were given a blank space like a little triangle or something plus something equals this, that's almost the same thing, and it just requires more thinking and more verbalizing since it is a word problem. I've had more success with word problems this quarter than I did in high school."

## Summary

Claire tended to apply negation to tasks with one unknown and compensation to tasks with more than one unknown. She was reluctant to accept hints on the polynomial expressions and algebraic fractions tasks but would readily accept them on the radical expressions and complex numbers tasks. Although she had demonstrated an instrumental understanding of multiplication and factoring, she still had difficulty connecting the two. She also had difficulty coordinating the conditions of multistep algorithms.

Claire would generalize her solutions to other specific instances but usually did not make any statement concerning her general process of solution. She was not consistent in her transfer of the processes and operations of polynomials to radicals and complex numbers. In these latter areas she lacked an intuitive feel for the nature of her solutions. She was often surprised by the answers she obtained.

Claire's initial perception was a strong influence on her solution attempt. Her solutions often included several errors, false generalizations, and false transfers. She was not confident in her ability to solve the radical expressions and complex numbers tasks.

## Patty

### Reversibility and Flexibility

Question 1. What evidence of a reversibility of algebraic operations is provided by someone who is successful in a basic algebra course?

Question 2. What evidence of a flexibility of approach to algebraic problems is provided by someone who is successful in a basic algebra course?

## A Pattern of Behavior

Patty was capable of solving problems with a variety of approaches, but tended to fix a particular approach as the appropriate one to apply to problems. For example, on the Generalization Test she demonstrated that she could quickly and accurately apply the rule for squaring binomials; yet on the Algebraic Interview Tasks she would consistently multiply binomials vertically. In commenting on the Generalization Test, she explained:

> I used to use FOIL in high school and in MATH 100 [my basic algebra course] he [the instructor] taught us to just put it like this [write down the binomials vertically] and then you multiply it backwards, like that way [starts on left] and that's the way I've been doing it. And so I don't use FOIL now I use multiplication because I seem to get it right that way and it makes more sense, for some reason.

Her reliance on the vertical form of the multiplication algorithm persisted in all four series of the Algebra Interview Tasks.

Patty's varied approaches to solving the Algebraic Interview Tasks are presented in Table 4.8. Though she tended to use negation first when encountering tasks in a new series, she was able to apply both types of reversibility. As indicated in the table, an interviewer provided instruction in the use of a compensation approach for a task on three occasions. She applied a compensation approach to all exercises of the type requiring the subject to find three algebraic expressions. She also used compensation in situations where she was required to find

## Table 4.8

Algebraic Interview Tasks Solved by Patty With Hints (Underlined) or Without Hints (Not Underlined)

by Negation (N), Compensation (C), or Another (A) Approach

(The number in parentheses indicates the order of presentation for the task.)

| | Find the Missing Algebraic Expression: | | | | | | Vocabulary and Algebraic Symbolism |
|---|---|---|---|---|---|---|---|
| | | One Expression ☐ * B = C or A * △ = C | | Two Expressions ☐ * △ = C | | Three Expressions ☐ * △ = ○ | |
| | - | 14(38,N) | - | 8(8,N) | - | 2(2,A) | 1(1,A); 3(3,A) |
| Polynomials | x | 12(12,N); 13(13,N*) | x | 9(9,N/C); 10(10,C) | x | 4(4,C); 11(11,A) | 5(5,A); 6(6,A) |
| | | | ÷ | 7(7,N) | | | |
| | + | 1(19,C); 5(15,N) | | | | | |
| Algebraic | - | 4(14,N*) | - | 12(36,C); 13(37,N) | | | |
| Fractions | x | 6(16,N*); 16(18,C) | | | | | |
| | ÷ | 7(17,C); 10(35,N) | | | | | |
| | + | 4(23,N) | | | | | 1(20,A) |
| Radical | x | 3(22,C); 5(24,N); | | | | | 2(21,A) |
| Expressions | | 6(25,N); 7(26,**) | | | | | |
| | ÷ | 8(27,C) | | | | | |
| | + | 1(28,N) | + | 7(34,C) | | | |
| Complex | - | 4(31,N&C) | - | 5(32,N&C) | | | |
| Numbers | x | 2(29,N); 3(30,N) | | | x | 6(33,C) | |

* Interviewer showed Patty a compensation technique
** Unable to solve with hints

two factors in a multiplication problem. She tended to use negation on all other algebraic tasks. After she was shown a compensation approach on Problem 6 of the Algebraic Fractions Series, she used that approach on the next several tasks and only stopped using it when she started on tasks from the Radical Expressions Series.

### Flexibility of Approach

Once Patty was comfortable with a task domain, she tended to use an approach that had been successful on preceding problems to attack a given problem. This approach involved little or no reflective thought for differences in problem types by operations but did involve a search for similarities in expressions. For example, she developed a negation technique for solving Task 1-7 (Find two monomials whose quotient is $-5AB^2C$):

> O.K., usually when I do something like this and I can't figure it out right away, I think of just numbers and I think, "O.K., find two numbers whose quotient is 5." O.K. 3, I mean 15 divided by 3 is 5. So if I, I guess what I do is just find . . . multiply this by something and then get the answer and then I'll have two monomials whose quotient is $-5AB^2C$.

Patty went on to choose an arbitrary monomial (3AC) and multiplied it by $(-5AB^2C)$. She was a little unsure of her technique up to the last step of her self-instituted check. When asked how she felt about her solution she replied, "I think I'm right. I don't know if I went about it the right way." To the interviewer's response that there was no right way, Patty retorted, "O.K., the best way." She applied this same technique immediately (without a numerical test) on her next task, Task 1-8 (Find two trinomials, $P(X,Y)$ and $Q(X,Y)$, such that $P(X,Y) - Q(X,Y) = -2X^2Y$). The next day she attempted to apply the same approach to Task 1-9 (Find two binomials, $M(X,Y)$ and $N(X,$[illegible] $= 6X^2 - XY - Y^2$). In this case the approach was inefficient and "became messy." Although she originally perceived the task as like the two preceding ones, she was able to break from this set, "Umm, O.K., wait a minute. Maybe if two binomials probably is like trying to factor that. And that would be two binomials, I think." She explained her reason for dropping her original approach in the following discussion:

> Because it was getting too messed up. And then I remembered there was an easier way to do it. And the reason I did it that way first is because that's what I did in the two before it. You know, it's that little trick, called mind set, or something like that.

At this point she switched approaches and solved the next two tasks (both involved finding two factors) by compensation.

Evidence of the flexible nature of Patty's ability to switch her flow of thought in the midst of solving a problem was provided in her solution to Task 2-1 (Find the algebraic fraction in lowest terms which when added to the sum of $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ yields $\frac{3X + Y}{Y}$) by adding $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$. Even while this process was going on, she began to suggest an easier way to solve the problem and eventually broke with her solution procedure to solve the problem using compensation.

Patty was not as comfortable working in the last two series of the Algebra Interview Tasks. The dynamic nature of the development of her flexible approach to new content domains can be traced through her solution to these tasks. Task 4-13 (Find the complex number which when multiplied by 3 - i equals 10) was solved by negation. First, she set up the equation $x(3 - i) = 10$. Next she divided both numbers by 3 - i and wrote $x = \frac{10}{3 - i}$. Even though she found that $(3 - i)(3 + i)$ equaled 10 in her next step, she continued the algorithm.

Upon solving the next task, she questioned whether there was not an easier way to solve these problems. When the interviewer asked Patty if she could think of any other way, Patty suggested a form of compensation but then responded, "This way at least I know that the answer is right." Then she solved Task 4-15 by negation. This time when she was asked by the interviewer if the problem could be solved another way, Patty suggested a compensation approach; though she was not convinced that it would work until she checked it. On the next task she suggested several solution approaches and was confident about the appropriateness of each approach.

## Reversibility of Multiplication and Factoring

Patty noted the relationship between factoring and multiplying polynomials on several occasions (for example, Tasks 1-4 and 1-9). In solving Task 1-11 (Write a binomial P(X,Y) and a binomial Q(X,Y). What is the product of P(X,Y) and Q(X,Y)? Next, take the product you've just found and factor it), she multiplied $3x + 2y$ and $2x - y$ and then stated, "You just multiplied the factors and got it so those are the factors." Yet in tasks such as Task 1-12 or 1-13 that required finding only one factor given the other factor she did not approach the problems through factoring or suggest it as an alternative method.

While trying to solve Task 1-13 (What trinomial multiplied by $A - B - C$ yields the product $A^2 - B^2 - C^2 - 2BC$?), she began by dividing $A^2 - B^2 - C^2 - 2BC$ by $A - B - C$. Though she indicated that she knew her approach would work, she dropped it to search for an easier solution. Her new approach was to write the indicated division as a fraction and to try to reduce the fraction by factoring an $A - B - C$ in the numerator and denominator. Even though this is somewhat circular (she was trying to reduce the fraction by finding what polynomial multiplied by $A - B - C$ was equal to $A^2 - B^2 - C^2 - 2BC$), she did not recognize that her new task was the same as the original problem. Instead she lamented that she could not cancel and sought assistance from the interviewer. Even though both tasks "got messy," she did not try to use compensation to find the missing factor.

## Reversibility of Multistep Algorithms

Patty had little difficulty with multistep algorithms. Her application of negation by choosing a value for an unknown and solving for the other unknown was well suited to problems with two unknown expressions such as Task 2-13 (Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$). In her solution, she did not refer to similar experiences in her class or during previous interview sessions. Instead she returned to using a numerical example and then solved the numerical example and Task 2-13 as she had Tasks 1-7, 1-8, and 4-4. The following excerpt from her solution to Task 2-13 demonstrated the internal nature of her solution technique for this type of item. After substituting a 3 for A and a 5 for B in the original task she restated the problem:

> Patty. I want two algebraic fractions in lowest terms with different denominators that will eventually equal 6/5. So . . . let me see . . . if you had . . . no I'm getting all confused . . . okay . . . alright, maybe I'll go, let's say that I have 6/5 and I'm going to add it to 4/7.
>
> Interviewer. Interesting, where did the 4/7 come from?
>
> Patty. Out of my head . . . I just made it up.

Patty went on to solve for the other fraction and then explained how she arrived at her solution:

> I just thought, well I was trying to go this way . . . I'm going backwards. I thought what minus what is 6/5? I'm going well 6/5 plus something equals something and when you subtract those two you'll get 6/5. So, the answer is not

> unique, because you can make up any number you want to . . . .. Now we have to do it with letters. Alright well this, now that I did it that way maybe it won't be so hard. 2A/B plus . . . I don't know . . . letters do it with letters . . . okay, plus C/D.

From this point Patty went on to finish her solution to the problem. Her realization that she could pick an arbitrary algebraic fraction and apply negation was transferred from her solution to the parallel numerical task.

## Ability to Generalize and Transfer

Question 3. What evidence of an ability to generalize algebraic process and operations is provided by someone who is successful in a basic algebra course?

Question 4. What evidence of an ability to transfer algebraic process and operations within the content of algebra is provided by someone who is successful in a basic algebra course?

Question 5. What evidence of the use of false generalizations and false transfers is provided by someone who is successful in a basic algebra course?

## Ability to Deduce the General from the Specific

Patty enjoyed responding to the challenge of whether her solution was the only possible one. In several situations where the interviewer did not ask, Patty would question, "Aren't you going to ask if my answer is unique?" In situations that require the student to find three algebraic expressions (see Table 4.8), she used the statement of the problem to describe the general nature of her solutions. On tasks that were solved by finding one unknown algebraic expression, she would focus on the uniqueness of the unknown. Even after stumbling through Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?), she was able to gain her composure and relate the task to numbers in order to respond to the questions of the uniqueness of her answer. Her solution to Task 1-12 was wrought with false generalizations (which are discussed later in this section). She spent a little over 12 minutes working on the task with hints from the interviewer. Her ability to rise above the frustration of the task and respond to the uniqueness of her answer was demonstrated in the following dialogue:

> _Interviewer_. OK, now is your answer unique?
>
> _Patty_. I hope so. What was my answer?
>
> _Interviewer_. Your answer was 16A . . .
>
> _Patty_. OK, alright, umm, yes. That doesn't sound very convincing, but it's not . . . you can't do anything else with that. I guess. What trinomial . . . I think that's unique.
>
> _Interviewer_. A little more convincing.
>
> _Patty_. That is unique! Because it's just like if you said this is 35 and that's 5, the only thing you can get is 7.

Patty finally convinced herself that her solution was unique by viewing it in a concrete fashion tied to her past experience. From this point on, she responded with conviction to the question of the uniqueness of her answers on tasks resolved by finding one algebraic expression.

The final type of problems included in the Algebra Interview Tasks involved situations of the form $\square * \triangle = c$ in which the subject was requested to find two algebraic expressions, $\square$ and $\triangle$, given an operation, $*$ , and the algebraic expression, c. She tended to be confident in her responses when the operation, $*$ , was addition, subtraction, or division. For example, after solving Task 2-12 (Create a subtraction problem involving two algebraic fractions in lowest terms with the same denominator and whose difference is $\frac{2A}{B}$) she was asked if her answer was unique. She responded,

> Is it unique? Naah! No, because you could just have any numbers there- it'd have to be "B." It'd have to be "B" on the bottom . . . because it has to be in lowest terms and B is about the lowest you can get there and you have to have the same denominator. So you couldn't have a different denominator and then subtract them and get "B." Well, I don't know, but it would have to be "B."

Her comments were exemplary of the general nature of her solutions to the addition, subtraction, and division problems in which she was requested to find two algebraic expressions. On multiplication tasks of this type she was a bit unsure of her response. She was generally not confident that the factors she found were unique and equally unsure of how to find any other possibilities. Her confusion with uniqueness in multiplication in terms of its distinction from the other operations was presented in her solution to

Task 1-9 (Find two binomials, $M(X,Y)$ and $N(X,Y)$, such that $M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2$):

> Well I don't know if I could do it another way. The other ones you could because you could pick any number you wanted to, like in the one that said find two terms whose quotient equals this. You could multiply the quotient by anything and get anything and those were the terms that you could use. Only this one, you know, I don't know if there's . . . how many factors there are. I guess if I sat here I.. maybe I could find two more . . . maybe . . . but I don't know . . . I'm trying to think, how else to find two other numbers. I mean, besides . . . you just factor it. I don't . . . I don't know.

Since there were no similar items in the other series of the Algebra Interview Tasks, it is impossible to infer whether she resolved her confusion about the nature of prime factorization as opposed to finding appropriate addends, or divisor and dividend, or subtrahend and minuend.

### Ability to Generalize from Task to Task

Patty tended to generalize a solution process from task to task within a given content domain. For example, she generalized her solution process for Task 1-7 (Find two monomials whose quotient is $-5AB^2C$) to her solution for Task 1-8 (Find two trinomials $P(X,Y)$ and $Q(X,Y)$, such that $P(X,Y) - Q(X,Y) = -2X^2Y$). Task 1-7 was her first experience with a task that requestd her to find two algebraic expressions. In order to solve the problem she first related it to working with numbers:

> O.K., usually when I do something like this and I can't figure it out right away, I think of just numbers and I think "O.K., find two numbers whose

> quotient is 5." O.K. 3, I mean 15 divided by 3 is 5. So if I, I guess what I do is just find . . . multiply this by something and then get the answer and then I'll have two monomials whose quotient is $-5AB^2C$.

On the next problem, Task 1-8, Patty generalized her previous solution involving two monomials and division to a task involving two trinomials and subtraction. This time she did not refer to numbers, but instead related the new task to its predecessor: "Oh, well if I wanted to do this one like the other one, I would . . . to find two such things I would take $-2X^2Y$ and add something to it and get an answer and I would add . . . I have to add a trinomial." As described earlier, she tended to try to extend a successful approach to each succeeding task. In this case her approach was a form of negation particularly well suited to tasks of the form $\square * \triangle = C$ where C was a given algebraic expression, * was either addition, subtraction, or division, and $\square$ and $\triangle$ were two algebraic expressions to be found. Her approach was to pick an arbitrary expression for one of the terms "to find" and then determine the other expression by using negation.

### Ability to Transfer

Patty transferred her application of negation to other content domains as well. For example in solving Task 4-5 (Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ such that their difference is $-2i$), she solved the task by first writing the equation $X - Y = -2i$ and solving for X. Then she substituted an arbitrary complex number for Y and solved for X. She compared her approach to techniques for graphing that she had recently learned in her basic algebra course: "This is just like . . . well we've just been doing slope and all that kind of stuff and you have equations. And you could make Y equal anything because then you just figure out what X is. So there's endless -- well I don't know if it's endless -- a lot that you could do." She compared her approach to solving the algebra task involving two terms to find to the graphing of linear equations in two unknowns.

Although she was able to transfer processes and operations from one series to another, the transfer was not always sure and definite. For example, she solved Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals -14?) by dividing -14 by $2 + 3\sqrt{2}$. In order to rationalize the denominator, she first suggested multiplying by $\sqrt{2}$. Next she tried squaring the term. Her procedure for multiplying $(2 + 3\sqrt{2})(2 + 3\sqrt{2})$ was the same one that she used for multiplying binomials. She wrote the radical expressions down vertically and multiplied "backwards"; i.e., starting from the left. After a hint by the interviewer focusing on the possibility of multiplying and getting the radical term to "add out," she suggested multiplying by $2 - 3\sqrt{2}$. She commented that this was something like multiplying $(A + B)(A - B)$ but did not apply this before being provided a hint.

In the next interview session Patty was asked to solve Task 4-2 (Find the complex number which when multiplied by 3 - i equals 10). The following excerpt from her solution reveals her lack of assurance in her transfer of the technique of rationalizing the denominator. First she set up the equation X(3 - i) = 10 and divided both members by 3 - i to obtain $\frac{10}{3 - i}$ as her solution. When the interviewer pointed out that Patty's answer might not be simiplified since it still contained a square root in the denominator Patty replied:

> Oh! Well, if I had a square root in the denominator, to get rid of it, I would multiply by . . . I don't know. Now the i confuses me. If that were . . . if that was the square root of something then I should multiply it by 3 + i but I don't know if that works. I could try it.

Patty was eventually directed to a solution of the problem and immediately generalized her solution in the next task.

Although Patty was hesitant in her solution to Task 4-2, she applied her approach smoothly in Task 4-3 (Find the complex number which multiplied by 5 - 3i equals 13 + i). Her solution included rationalizing a denominator, three instances of multiplying complex numbers as she had binomials (with the added steps of substituting -1 for $i^2$ and simplifying), and reducing an algebraic fraction. After solving the problem and checking her work she exclaimed, "That's amazing! I'm really surprised. I wonder is there some easy way that I could do that?" The interviewer suggested that the task was "something like factoring," but

Patty, although acknowledging that it was indeed like factoring, stated that she did not see the relationship.

Patty's inability to transfer her experience in factoring polynomials to complex numbers was further demonstrated in her solution to Task 4-6 (Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$, such that their product is a negative integer). First she set up the equation $(X)(Y) = -Z$ and planned:

> I think what you have to do is . . . first I was thinking okay you want to get . . . if you're multiplying two of these together . . . . Okay, you don't have to worry about the i's, these two i's because when you multiply these two together it's going to turn out to be -1 and you're going to get rid of those i's but you want to make sure that they cancel out . . . I know what I'm trying to say. Alright, I'm just going to try two numbers so I can see what I'm doing.

Not only did Patty have difficulty stating her plan, but her choices of numbers revealed a lack of connection with her experience in factoring. The sequence of her search for possible numbers was $(3 + 4i)(2 - 3i)$; $(2 - 4i)(2 - 4i)$; $(2 + 4i)(2 - 4i)$; $(-2 - 4i)(2 + 4i)$; $(-2 - 4i)(-2 + 4i)$; and $(4 - 2i)(4 + 2i)$. At this point the interviewer directed Patty to a solution. Although her plan suggested a focus on the difference of two squares as a way to eliminate the imaginary term, it was not until her third pair of complex numbers that she actually tried that possibility.

Patty was unsure of the transfer of addition and subtraction of polynomials to addition and subtraction of complex numbers. On Task 4-1 (Find the complex number which when added to $-3 + 4i$ equals $2 + i$), she was hesitant with the use of i even though she had "sort of" used it before. As she stated, "I don't know. This 'i' business throws me off. Am I supposed to treat it just like a variable right now or am I supposed to treat it as . . . what is it, it's the $\sqrt{-1}$?" She went on to solve the problem, but did not realize what her solution would be like as she gingerly went through it. Although it was not clear to her during the solution of the problem, she noted after solving it, "I was lucky because I didn't have to use whatever the heck i is!" Later in the same interview session she added and subtracted complex numbers in Tasks 4-4 and 4-5 with no apparent difficulty both applying negation and compensation techniques.

In the next interview session the confusion again rose in Patty's solution to Task 4-7 (Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ such that the sum of the two complex numbers is i). The confusion that i created emphasized the role of perception in a student's ability to transfer a process from one content domain to another. As she stated, "I'm never too sure about these i's." The confusion created by the complex numbers in Task 4-7 was a major contributor to her attempts to add 1/3 i, $-3i$, and $-4i$ in turn to $3i$ in an effort to obtain $1i$. Further examples of the effect of confusion on her solution process are provided in the next section.

## Errors, False Generalizations, and False Transfers

The greater difficulty of the addition or subtraction of complex numbers as compared to the addition or subtraction of polynomials hinges on the student's perception of complex numbers. How comfortable is a student with the meaning of a complex number? Mathematically, it is just as easy to add $3i + 4i$ as it is to add $3X + 4X$ or $3\sqrt{5} + 4\sqrt{5}$. Patty had previously demonstrated an ability to add polynomials and radical expressions. The added complexity of working in a domain that is not totally meaningful might be described as psychological.

Another example of psychological complexity is the situation where a perceived "lack of closure" leads a student to make errors. Task 3-4 (Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$) was designed to produce a feeling of "lack of closure." Although Patty was frustrated by the incompleteness of her response, $\sqrt{6} - \sqrt{3}$, she did not apply a false generalization to resolve her frustration. This is not meant to imply that she ever viewed her answer as complete, but that she was unwilling to create a false rule to cover the situation.

There were, however, several false generalizations firmly established in Patty's memory. These generalizations tended to recur in solving tasks in which her focus was drawn from checking the appropriateness of each step of an algorithm by the mathematically complex nature of the task or by the relative ease of the task. One example of a recurring error that was evidently triggered by the application of an algorithm with a minor "bug" as an initial step was her multiplication of the first term in a product of binomials. It made little difference whether she was applying a given rule, as on the Generalization Test, or multiplying the binomials vertically; her error pattern was stable. Her error on the Generalization Test, described earlier, also occurred during her application of a vertical algorithm for multiplying binomials in her solutions to Task 1-9 and 1-11. On this task she multiplied $(-3X + Y)(2X - Y)$; $(3X - Y)(2X + Y)$; and $(3X + Y)(2X - Y)$ and in each case wrote the first term of her product as a linear term. Even though the task requested two factors whose product was $6X^2 - XY - Y^2$ and she had obtained $6X - XY - Y^2$, she did not notice her error. On Task 1-11 she again multiplied binomials erroneously:

$$\begin{array}{lllll}
3X & + 2Y & & & \\
\underline{2X} & \underline{- Y} & & & \\
\hline
6X & + 4XY & & & \\
 & - 3XY & - 2Y^2 & & \\
\hline
6X & + XY & - 2Y^2 & &
\end{array}$$

This time the interviewer pointed out that Patty had made an error, and Patty asked to find it herself:

> What could it be? Oh, I know what it is. No, that's not what it is. Yes, that is what it is. Is that what? [Patty corrected the error and interviewer affirmed the correction.] I know why that is. It's because I get mixed up between just multiplying and adding. Cause you know when you add it's just . . . . you're adding 3X + 2X it's just something X, right? And then when I multiply I forget that it has to be X . Of course, if I thought about it then I would remember.

In Patty's algorithm the error was made consistently on the first term, but not on the last term. Like a phonograph needle touching a record, the effect of starting a procedure provided a chance for error until the groove was caught.

In the above situation the problem was not difficult for Patty. In such cases if she did not maintain her attention on her tendency to err she would tend to falsely apply her multiplication process. In complicated situations involving variable exponents her difficulty with knowing when to add and when to multiply terms of algebraic expressions was heightened. The following excerpt from her solution to Task 1-6 (Simplify the following expressions by finding the indicated products and combining similar terms: A) $X^n \cdot X^n$; B) $5X^n(6X^n + 3X^2 + 2X^{n-1})$; C) $(5X^n + 2)(6X^n + 3)$; and D) $(X^n + Y^{n-1})^2$) emphasized this difficulty:

> Patty. "Simplify the following expressions by finding the indicated products and combining similar terms."
>
> Interviewer. Read the first problem.
>
> Patty. Okay. "A) $X^n \cdot X^n = X^{n+n}$" and I think "I hope that's right." "$5X^n(6X^n + 3X^2 + 2X^{n-1})$." Okay. First I do what's in the parentheses . . . $6X^n + 3X^2 + 2X^{n-1}$ . . . Umm, there's no like terms I can combine in the parentheses so I'll just multiply each one by $5X^n$. So I'll have $(5X^n)(6X^n)$ is, wait a minute . . .
>
> Interviewer. Tell us what you're thinking.
>
> Patty. Okay, I get confused whether it's, if the n just stays the same, or if it's n + n,umm . . .
>
> Interviewer. Tell us what you're thinking.
>
> Patty. I'm trying . . . I can't even think straight. I'm trying to think of it in numbers

> now. So if I have $3X^2$ times something else $X^2$, it's X to the [fourth]. Okay. So I do . . . it's $3X^{n+n}$. I think. I really don't know. Umm . . . + $(15X^n)(3X^n)$ that's $15X^{n+2}$ + $(5X^n)(2X^{n-1})$ is $10X^{n+(n-1)}$. And there aren't any more similar terms so I could just leave it like that or I could factor out. No, if I did that, that would be dumb because then I'd just get back to that. So I guess that's what I should leave it as. The next one is "$(5X^n + 2)(6X^n + 3)$," and if I want to combine like terms I could make it $(5X^n + 6X^n)$ $(2 + 3)$ and I have $(30X^{n+n})(6)$. And that means that I have $180X^{n+n}$, I think. Okay. The next one is $X^n + Y^{n-1}$ and both of them to the second $[(X^n + Y^{n-1})^2]$. Which is like multiplying $(X^n + Y^{n-1})$ times itself and so . . .
>
> Interviewer. What are you thinking?
>
> Patty. What I'm thinking is that in this one,
>
> Interviewer. Which one?
>
> Patty. C, I'm not sure if I was allowed to switch around the 6X and the 2. Well I'm just going to multiply this one the way I learned to do it in class, which is you put one on top of the other and you multiply them. And you get $2X^{n+n} + X^nY^{n-1} + X^nY^{n-1} + 2Y^{(n-1)+(n-1)}$.

With hints from the interviewer Patty resolved her last two problems. Although she left the problem with all parts correct, she relied on authority (how she learned to do it in class) for her solution to part C and was never convinced of why her first method was inappropriate. Just two weeks earlier she had obtained a 97 on her test on operations with polynomial expressions in the basic algebra course. Her score on the test does not reflect her difficulty with the meaning of the processes that she applied.

Patty also had trouble with the rules for multiplying radicals with different indices. Her errors on Tasks 3-7 centered on when it was appropriate to add indicies.

Although the interviewer finally resorted to directed instruction to try to resolve her difficulties, the interviewer was unable to lead Patty to a solution, and the problem was never solved. Her intuitive feeling that a square root times a cube root should be a fifth root recurringly surfaced during her solution.

Patty's battle with inappropriate cancelling in division was her most consistent source of difficulty on the Algebra Interview Tasks. It appeared in five of her solutions from three different series. In some of these tasks she noted her own error, though she was unsure why she was wrong. For example in solving Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?), she wrote $\frac{64A^3 + Y^3}{4A + Y}$. Then she stroked out the Y in the denominator and wrote a $Y^2$ above the Y in the numerator. Suddenly she stopped and questioned, "No, what happened?" After a pause the interviewer questioned what Patty was thinking, and she replied, "I'm thinking, 'Remember Patty before you cancel you have to think about it,' and these are adding and they're not multiplying, so I don't think you can just go and do that. So maybe, I'll do this the long way to make sure I do it right."

Again while solving Task 1-13 (What trinomial multiplied by $A - B - C$ yields the product $A^2 - B^2 - C^2 - 2BC$?), Patty lamented the fact that she couldn't just cancel. After writing the problem as $\frac{A^2 - B^2 - C^2 - 2BC}{A - B - C}$ she noted:

> Now I can't . . . well maybe if you divide this . . . both of them by that. Factor out A - B - C . . . I don't think you're allowed to do that. Then you would get A - B - C . . . multiplied. So I don't think you can . . .I know you can't just cancel them out. I'm not sure why, but you're not allowed to do that.

In the above situations Patty wrestled with herself over the appropriateness of her false generalization. In her solutions to Tasks 1-9, 2-10, and 3-5 her errors went unchallenged until pointed out by the interviewer. In trying to solve these tasks her attention was drawn from her cancelling process by the complex nature of the task. For example, in solving Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals -14?), she reduced $\frac{42\sqrt{2} - 28}{-14}$ by dividing -14 into 28. The intuitive nature of this error keeps it readily available in memory for application in any division of algebraic expressions written as a fraction. Instruction in class and during the interviews had little effect on diminishing its stability.

## Related Abilities, Attitudes, and Perceptions

Question 6. What evidence of the role of perception and memory on the solutions of algebraic problems is provided by someone who is successful in a basic algebra course?

Question 7. What evidence does someone who is successful in a basic algebra course provide of her attitude towards algebra and her perception of her understanding of algebra?

## Perception and Memory

Patty's initial perception and memory for algebraic problems was not crucial to her solution of problems. Her flexibility enabled her to switch from one solution process to another, or from one generalization to another, in the midst of solving a problem. Hence after her initial perception she would pull from memory relevant bits of information and solution procedures, but as new perceptions were made and additional information obtained from memory, she would adapt her procedures. This adaptive process might be applied "on the spot" or occur after a lapse in time. An example of an "on the spot" adaption of perceptions was provided in her solution to Task 1-6, part B (Simplify the following expressions by finding the indicated products and combining similar terms: $5X^n\ (6X^n + 3X^2 + 2X^{n-1})$. After multiplying, she looked at her answer to see if it was in simplest form. While thinking of writing $30X^{n+n} + 15X^{n+2} + 10X^{n+(n-1)}$ in a simpler form, she thought of factoring. Immediately she reacted to the incorrectness of that suggestion. As she stated, "And there aren't any similar terms so I could just leave it like that or I could factor out. No if I did that, that would be dumb because then I'd just get back to that." She was able to evaluate her perception in terms of the original task and its effect on the solution of the problem.

In solving other problems, Patty's incorrect application of a procedure from memory based on her perception of the task created errors she did not detect. In her solution to Task 1-12 (What trinomial multiplied by $4A + Y$ yields the product $64A^3 + Y^3$?), she used long division, with some difficulty, to obtain an answer for the task. As she attempted to check her work an interesting switch in processes occurred. Rather than add partial products, she subtracted. Two possible sources for the error were her confusing the division and multiplication algorithms and her desire for terms to drop out of the product of $4A + Y$ and $16A^2 + 4AY + 4Y^2$ (her incorrect solution to the long division problem). She had difficulty finding her error, and the interviewer questioned, "I don't understand why you subtracted here." Patty responded, "Umm, now wait. Oh, I'm supposed to be adding. Why I'm getting this confused is . . . I'm getting this confused with something . . . You're right . . . that's what . . . I'm supposed to be adding." Later in the interview, while she was checking her steps in her application of the division algorithm, she realized the probable source for her error:

> All right, $4A$ goes into $64A^3$, $16A^2$ . . . Okay . . . so we multiply this by that and we get that. And we multiply the Y and we get $16A^2Y$. All right, and the way my professor does it is usually he moves that way over there, if you know you're going to have another term. So I would . . . Oh, I know what I did wrong . . . That's why I subtracted there cause I was thinking of subtracting . . . I couldn't figure out why I did that. Oh good.

In her solution process, Patty may have natural points at which a process is checked. Once a process begins, it is not interrupted until after it reaches the check point -- like completing one sentence before beginning a second. This may help to explain why she would continue a process after she had already obtained the answer to the original question. The original question was no longer at issue; instead her process needed resolution. Once it was resolved, she would note that her solution had to be correct because she already had checked it while solving the problem. For example, as a step in her solution to Task 4-2 (Find the complex number which when multiplied by $3 - i$ equals 10) she multiplied numerator and denominator of the fraction $\frac{10}{3 - i}$ by $3 + i$. In doing so, she found that $(3 - i)(3 + i)$ equaled 10. This went unnoticed until, after several steps she had obtained $3 + i$ by reducing $\frac{30 + 10i}{10}$. At this point she noted that $3 + i$ must have been right because she had just multiplied $3 + i$ and $3 - i$ and got 10. The same situation arose in her solution to the related Task 3-5 (What number multiplied by $2 + 3\sqrt{2}$ equals $-14$?).

As the basic algebra course developed, Patty changed the way she tended to set up a problem situation. Tasks from the first series were set up as algebraic expressions. Tasks from the algebraic fractions series were represented as equations with either a box or a gap representing the "unknown." The radical expression and complex number tasks were represented as equations with the unknowns represented by literal variables. Although this variation may have resulted from her concurrent instruction in basic algebra, she was consistent in her classification during the "catch up" session in which she solved tasks from three of the four series. During that interview, she initially represented Task 4-7 as an equation with literal variables; Tasks 2-10, 2-12, and 2-13 as an equation with gaps or boxes; and Task 1-14 as an algebraic expression.

### Attitudes and Self-Perceptions Concerning Algebra

Patty's perception of her knowledge and processes in algebra appeared to be affected more by her failures than her successes. Upon entering the course, this was no doubt a natural reaction to having failed the exemption test and being required to take basic algebra. But as the course progressed, her self-perception seemed unaltered by her success in basic algebra as measured by her course tests.

Patty felt unsure of herself in situations in which she did not have a ready process or solution. In solving Task 1-4 (Find two binomials whose product is a binomial. Find the product) she noted that since her product had to be a binomial the rule $X^2 - Y^2 = (X + Y)(X - Y)$ applied. She explained her solution as follows: "All of a sudden I thought of all types of ways you learn to factor, and I thought of $X^2 + 2AB$, or I mean $X^2 + 2XY + Y^2$, and I said no you can't use that." Because of her apparent reliance on special cases for factoring to solve the task, the interviewer followed up this task by asking Patty to find two binomials whose product had four terms. She explained the difficulty she perceived the change would create:

> Umm,umm . . . See now I get stuck because I have no formula in my head for that. This is easy because I know that the little formula will come out to be two things and I'm going to have to sit here and experiment for hours trying to find one that's a quadranomial. Two binomials that multiply to give you four terms . . . Umm . . . I have no idea where to start. Maybe if you have more than two variables . . . (XY + Z)(A + B). God, that would be weird! Ha,ha, ha. Umm . . . Can I sit here and try things 'til I get it cause I can't think of any little system to do? Well, let's see . . . I'll just do that and see what happens. I mean who knows what's going to happen.

Patty did not suggest that her solution was correct until she actually multiplied her binomials and tried to combine similar terms. Without a special rule in memory, she perceived that she would need to solve the problem by trial and error and had little feel for where her trial would lead.

In terms of her solution process to algebraic problems in general Patty noted her difficulty with making careless mistakes and with changing signs. After completing Task 1-12 she lamented, "I just make careless mistakes so many times . . . I always make them . . . I just was not paying attention to what I was doing." She perceived one source of her careless mistakes to be confusion with when to add exponents and when to leave them alone. She noted that another source of careless mistakes was her confusion with signs. For example, she stated that her initial reaction to Task 1-13 was, "Oh gosh, you have to remember all this stuff about signs changing when you multiply." Then after solving Task 4-5 with help from the interviewer, she complained about the problems created by her careless mistakes, "Those are the things that mess me up is that I forget to do things like that [change the signs when she divides by a negative] or I make . . . you know signs have always messed me up."

Patty's attitude toward algebra was centered on specific content or operations. In reaction to Task 1-7, which she solved by multiplication, she expressed a strong dislike for division: "I hate things like these because I hate division!" Radicals were also disliked. Despite her confusion with complex numbers she viewed them positively: "I didn't mind those [the complex numbers tasks] at all really because they were interesting. We just skimmed over those in class and I just didn't understand them really. So now I kind of understand them." Some of her other attitudes toward algebra are presented in the following discussion of her completion of the Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test.

### Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test

Patty's classifications for the 21 tasks of the Classification Test are presented in Table 4.9. She separated the cards into three piles. The first pile contained those tasks in which, according to Patty, "They just told you what to do to it." She noted that each card

Table 4.9

Patty's Solutions for the Classification Test and the Preference for Algebraic Problems Test Coded by Type of Algebraic Expression - Polynomial (P), Algebraic Fraction (F), Radical (R), Complex Number (C); Operation - Add (A), Subtract (S), Multiply (M), Divide (D), Factor (F); and Style of Presentation - Textbook (T), One (1), Two (2), or Three (3) Missing Expressions to Find

CLASSIFICATION TEST

PREFERENCE FOR ALGEBRAIC PROBLEMS TEST

| | SIMPLIFY | FIND A MISSING NUMBER | CREATE | |
|---|---|---|---|---|
| 1 | P-F-T | | | 1 |
| 2 | C-M-T | | | 2 |
| 3 | C-A-T | | | 3 |
| 4 | P-S-T | | | 4 |
| 5 | | C-S-1 | | 5 |
| 6 | R-A-T | | | 6 |
| 7 | F-S-T | | | 7 |
| 8 | | | P-D-2 | 8 |
| 9 | | F-D-1 | | 9 |
| 10 | F-D-T | | | 10 |
| 11 | | F-A-1 | | 11 |
| 12 | | | P-M-2 | 12 |
| 13 | P-M-T | | | 13 |
| 14 | | P-S-1 | | 14 |
| 15 | | R-D-1 | | 15 |
| 16 | | R-M-1 | | 16 |
| 17 | | | F-S-2 | 17 |
| 18 | | R-A-1 | | 18 |
| 19 | R-D-T | | | 19 |
| 20 | | | P-M-3 | 20 |
| 21 | | | C-M-3 | 21 |

said "simplify" except one that said "factor." The second pile was categorized by Patty as tasks in which the subject was "requested to find one missing number." "Number" was used loosely here to include all algebraic expressions. The third pile she distinguished by each task requesting the subject to create a problem. As she described the tasks, "I think it was just that I could sit there and make up my own thing as long as I followed the rules that you gave."

Based on Patty's classification scheme her categorization of Task 1-9 is interesting. Although the solution to the task is unique, as normally viewed in basic algebra, she placed it in the "create" category. Yet the factoring task was included in the "simplify" category. Perhaps the verbal statement or algebraic symbolism of Task 1-9 acted to distinguish the task from factoring.

Patty's solution to the Mathematical Memory Test was limited by the manner in which she completed the Classification Test. She explained, "When I went through looking for cards like the #1 card I noticed that there were three types so after that the next one I really didn't look at the #2, I just thought in my mind what type it was. And I did that for the third one." She circled a 2 on an arbitrary card but had already decided how to classify other cards like the circled one, so she didn't notice what it said. She was unable to provide any detail for the cards with a circled 2 or 3 in the right hand corner. Since she did read the card circled 1 prior to making her classifications, she was able to recall more about that task:

> Oh, the first one that you circled was a subtraction problem between two algebraic fractions and it wasn't totally factored yet or it wasn't in simplest form or whatever. And you said simplify. The first fraction was still factorable in the top part. I think it was a trinomial in the top. I think it was something $X^2$ + something X + something and the denominator of the first fraction I think was like X - 7 or something like that . . . The second fraction was reversed. The bottom was still factorable and the top was just whatever.

Patty's description was initially correct. She remembered the content domain and the operation for the task. Beyond that it is possible that she became confused with other bits of information in memory. For example, her description of the details of the problems was very much like the other algebraic fractions task included in her "simplify" category.

After the cards were reshuffled, Patty arranged the tasks in the order in which she felt most comfortable in solving them. Her order of preference is presented in Table 4.9. She commented that the factoring task was chosen first because vocabulary "bothered" her. Complex number tasks were included early in her inventory because: "Since doing all those complex numbers it doesn't bother me anymore."

Patty tended to group cards together in her ordering on the basis of algebraic content domains and style of format for presentation of the task. Fraction tasks were generally grouped in the midrange of the inventory. These were followed by tasks involving radicals. She explained this tendency as follows:

> I hate terminology like real numbers, integers and all that stuff. I also don't like square roots very much. This one [#18] I hate! I hate this one [#19] for some reason. I looked at it and I said, "I just never remember what to do," cause I wouldn't remember how to rationalize the denominator. I wouldn't remember if I just multiply it by $\sqrt{3}$ or if I have to multiply it by $1 + \sqrt{3}$ or whatever I have to do. I just decided I hated it.

Patty's last three selections contain an interesting common feature that she noted she had not realized consciously when she ordered the tasks. A key step in solving each of these tasks is based on the application of the generalization for a difference of two squares.

Patty's preference for nonverbal tasks is an example of the relationship between her classification of the 21 algebraic tasks and her preference for solving these tasks (see Table 4.9). Of the nine tasks she classified as problems "to simplify," seven appeared in the ten most preferred problems to solve. Her second classification type tended to fill the middle ranks of the preference inventory. Of the seven tasks she classed as problems "to find a missing number," five appear in the range 5-15 of the preference inventory. Finally, her third classification type tended to be at the end of the preference inventory. Of the five tasks she classified as problems "to create," three were included in the last five tasks of the inventory.

Apparently, she classified the tasks into categories similar to her perceived difficulty with the tasks.

## Summary

Patty was generally flexible in her approach to algebraic problems. She was able to break from a procedure and switch to an alternative approach. She was also able to solve a problem in more than one way. She was comfortable applying both forms of reversibility, though in new content domains she was most comfortable initially with negation.

Patty was able to extend a class of appropriate solutions to a general solution technique. She transferred processes from arithmetic to algebra and from polynomials to radicals and complex numbers, but was at times unsure of the correctness of the transfer.

Patty applied false generalizations with a fairly high degree of regularity. As she put it, "I always make careless mistakes!" The false generalizations and false transfers were applied in all series of the Algebraic Interviews.

Patty's solutions were often guided by her perceptions of the task. Her flexibility and ability to generalize were activated by a continuing input of perceptual cues. These cues encouraged the application of false generalizations as well as correct ones.

# CHAPTER V

# OBSERVATIONS ACROSS SUBJECTS

In this chapter the data presented in the four case studies of Chapter IV are analyzed across the subjects. Regularities and differences between the subjects are presented. First, the subjects are described in terms of the information available on their backgrounds, and their performance on the CSMS Algebra Test and Generalization Test. Next, four processes of the understanding of algebra (reversibility, flexibility, the ability to generalize, and the ability to transfer) are discussed in terms of the common features observed in the students' solutions to the Algebra Interview Tasks. Finally, some related processes, perceptions, and attitudes observed in the students' solutions to the Classification, Mathematical Memory, and Preference for Mathematical Problems Tests are examined.

## A Description

### Background Information

Since the purpose of the study was to investigate several processes of understanding of students who were successful in a basic algebra course, several similarities in the subjects' backgrounds resulted from the selection procedure. To be included in the study, the subjects had to have taken at least two years of algebra in high school and to have been rated by their basic algebra instructor as above average. This was done to increase the odds that after they completed the basic algebra course they would still be viewed as successful. To be viewed as successful in their basic algebra course, the subjects were required to have at least a B average on their basic algebra tests covering the topics used in the study.

In addition to the similarities created by the selection procedure, all subjects had at least a 3.0 cumulative grade point average in high school, were enrolled in the College of Arts and Sciences, had above a 2.5 cumulative grade point average at the University of Georgia, had failed the University of Georgia's exemption test for basic algebra, and received at least a B as their final grade in basic algebra. Other commonalities (found in the backgrounds of at least three of the four subjects) included quarterly grades of A or B in all high school courses except mathematics; completion of a year of geometry and at least a quarter of another high school mathematics course requiring algebra as a prerequisite; and a score of at least 540 on the verbal component of the SAT examination. Perhaps the most notable difference in the subjects' backgrounds was in their scores on the mathematics component of the SAT examination. These scores ranged from 410 to 610.

## CSMS Algebra Test

The CSMS Algebra Test was used to describe the students' understanding of algebraic symbolism. The subjects experienced little difficulty with the six variants for the meaning of literal variables measured by the CSMS Algebra Test. All of the subjects had at least 80 percent of the 51 responses correct. At least three of the four students made errors on items in which letters were used as a specific unknown, used as a generalized number, and used as a variable. Their common errors occurred on tasks necessitating the coordination of more than one condition for a correct solution. For example, three students used only one of the given conditions in attempting to solve Item 14 (What can you say about r, if $r = s + t$ and $r + s + t = 30$?).

The errors the subjects made were not consistent with either Küchemann's (1978) classification scheme for the different ways letters are used in algebra, or with his analysis of items in a Piagetian framework. Students' errors tended to be scattered throughout Küchemann's classification scheme and did not support his analysis of the levels of operational thought necessary for the solution of particular tasks. For example, one student correctly answered all four items listed by Küchemann as requiring a late formal stage of development for solution. Yet, she made errors on three of the eight items listed by Kuchemann as requiring a late concrete stage of development for solution.

### Generalization Test

The common behavior observed in the subjects' solutions to the Generalization Test was their apparent ease in generalizing the first rule, $(a^m)^n = a^{mn}$, to all of its variants. The subjects varied in their success in generalizing the other rules to their variants. Three subjects developed specific "bugs" in their application of Rule 2, $(A + B)^2 = A^2 + 2AB + B^2$. For two subjects the "bug" involved an incorrect application of Rule 1; i.e., when a variable factor of a term had a numerical exponent greater than 2, the subjects applied the false generalizations $(a^m)^n = a^{m+n}$ and $(a^m)^n = a^{m^n}$.

These same two subjects had difficulty applying Rule 3, $(a,b) + (c,d) = (ad + bc, bd)$, because of its syntactic similarity to the rule for multiplying binomials and the associative property. They also experienced some minor confusion in being presented mixed applications for Rule 3 and Rule 4. Their performance on tests on polynomials and algebraic fractions was inconsistent with the errors they made on the Generalization Test. The subjects had scored 93 and 94 out of 100 on the test on polynomials and 100 and 99 out of 100 on the test on algebraic fractions.

## The Questions of Interest

### Reversibility and Flexibility

The analysis of the case studies suggests that students can be characterized by their ability to use compensation or negation on appropriate algebraic tasks. Further, there are wide differences in their ability to switch from one approach to another approach even with specific hints.

A pattern of reversibility. Each subject was unique in the pattern of reversibility she applied to the Algebra Interview Tasks. The patterns ranged from the nearly complete dominance of a particular approach, either negation or compensation, to a flexible application of both approaches and a tendency to use negation in selected content areas.

Flexibility of approach. The subjects' patterns of reversibility were also reflected in their ability to switch from one approach to another. Those subjects with a tendency to approach problems by either compensation or negation, but not both, had difficulty solving the tasks with the other approach, even with very specific hints. Only one of the four subjects was able to solve the tasks in more than one way. She would at times select her approach in advance, based on the statement of the problem. She was also flexible in following hints from the interviewer even when the hints suggested a switch in approach.

Reversibility of multiplication and factoring. All subjects noted the relationship between multiplication and factoring, but they varied in their use of the realtionship. The only subject to use the relationship consistently in tasks of the form $a * \square = c$ or $\square * b = c$, where a, b, c are given algebraic expressions and $\square$ is an unknown algebraic expression, solved almost all tasks by compensation. The other three subjects consistently used the relationship between multiplication and factoring in tasks of the form $\square * \triangle = c$ where c was a given polynomial and $\square$ and $\triangle$ were unknown polynomials, but they tended to use negation to approach other tasks in which the relationship between multiplication and factoring could be applied.

Reversibility of multistep algorithms. The subjects' ability to reverse multistep algorithms was dependent on their style of approach to the algebraic interview tasks. The application of negation to multistep tasks with more than one unknown required the subject first to choose an appropriate algebraic expression for all but one of the unknowns and then to solve for the missing unknown. This procedure was used by two of the subjects. The application of compensation to multistep tasks with more than one unknown required the subject to work backward from the solution, note possible intermediate steps, and select appropriate algebraic expressions to yield the intermediate solution. For example, a student had to realize that the difference of two fractions may first be written with a common factor in the numerator and denominator and then simplified. One of the subjects experienced difficulty in applying either compensation or negation to multistep tasks. Very specific hints were required to guide her to a solution. The subject who was flexible in her use of compensation and negation applied the most efficient approach for each of the multistep tasks. The two subjects with a definite preference toward either compensation or negation continued to apply their preferred approach to all tasks involving multistep algorithms, even on tasks more easily solved by using the other approach.

In summary, these students who were successful in a basic algebra course had very different and definite styles for solving the algebraic interview tasks. Only one subject had complete reversibility of algebraic operations. Another readily applied compensation to the tasks, but was unsure of the appropriateness of applying negation; if a task indicated addition as the operation she seemed uncomfortable using subtraction to solve it. A third subject readily applied negation, but would only awkwardly and with hints apply compensation to tasks other than those involving multiplication. The fourth subject allowed the interviewer's hints to guide her application of negation and compensation for many of the tasks.

## Ability to Generalize and Transfer

### Ability to generalize process from task to task.

All of the subjects demonstrated that they generalized processes used in solving the Algebra Interview Tasks from one task to another. This was consistent with their success in basic algebra.

### Ability to deduce from specific to general.

On tasks with more than one solution, all of the subjects provided evidence of their ability to extend their class of solutions to other, more complicated variants of the tasks. Two subjects tended to solve these tasks by negation. For example, on a task of the form $\square * \triangle = \bigcirc$, where $\square$, $\triangle$, and $\bigcirc$ are unknown algebraic expressions and $*$ is a given operation, they would first make up two of the unknown algebraic expressions and then apply negation to solve for the third algebraic expression. They would extend their class of solutions in general terms; i.e., they would describe the conditions necessary for their algebraic expressions to be appropriate and the process they would use to determine other variants. The case studies of the other two subjects provided little or no evidence of their ability or inability to express a set of solutions in general terms.

Ability to transfer. Although all subjects recognized the relationship between the processes and operations of arithmetic and the processes and operations of algebra, only three subjects initiated the use of arithmetic variants of the algebraic tasks as a heuristic for solving the tasks. The fourth subject used arithmetic variants only if she was directed to by the interviewer. Two subjects experienced difficulty in transferring processes from arithmetic to algebra. For these subjects the transfer was inhibited by the added confusion of the use of variables, exponents, and polynomials.

Evidence of the transfer of processes and operations from polynomials to radical expressions and complex numbers was provided by two subjects. These subjects referred back to processes such as combining similar terms and multiplying two binomials to obtain a binomial in their solutions to tasks involving radical expressions and complex numbers. The other two subjects were unsure of the appropriateness of this transfer. One of these subjects needed to be repeatedly reminded of the relationship between the addition and subtraction of complex numbers and combining similar terms in order to solve the complex number tasks. And both subjects had difficulty determining what to multiply by the denominator of a fraction involving radical expressions or complex numbers in order to rationalize the denominator.

Errors, false generalizations, and false transfers. Although the subjects were successful in a basic algebra course, their protocols provided evidence that they had not completely eliminated incorrect variants of the processes and operations they perform. When the subjects were unsure which process to apply they would at times generalize or transfer incorrect processes. For example, while working with multiplication of polynomials with variable exponents, all of the subjects indicated some confusion over whether to add or multiply exponents. This confusion again arose when multiplying radicals with different indices. Three subjects wrote the radicals with rational exponents, disregarded the necessary condition of having the same base, and then multiplied exponents instead of adding exponents. Two subjects "cancelled" in situations where they viewed their fractions as indicated division. For example, $\frac{28 - 42\sqrt{2}}{-14}$ was simplified by dividing the 28 by -14. These false generalizations and false transfers were generally made on tasks the subjects viewed as relatively complex. Although all subjects experienced a feeling of "lack of closure" with Task 3-4 (Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$) and suggested the application of false transfers to provide closure, only one subject applied a false transfer in her solution. The lack of closure was not sufficient to make the subjects generate false generalizations and transfers.

In summary, these students who were successful in a basic algebra course demonstrated an ability to generalize algebraic processes and operations from task to task, but differed in their apparent ability to generalize general solutions from specific instances of a class of solutions and in their ability to transfer algebraic processes and operations within the context of algebra. Students who were successful in a basic algebra course generalized and transferred erroneous processes and operations that had been apparently extinguished in their work with simple polynomial expressions.

## Related Abilities, Attitudes, and Perceptions

The subjects' initial perception of a task proved to be a strong influence on their successful solution. For three subjects the perception of the task set a solution process in motion that was rarely broken until it was either completed or they were unsure of how to procede. Even though an indicated division might be represented as a fraction, if it was seen as a division problem, the rules for simplifying fractions were not adhered to; operations were completed after the solution to an original problem was obtained; and two subjects went on to factor the product of two polynomials they had been asked to multiply. Even though the subjects might effortlessly follow the algorithms suggested by their perception of the task, they were at times surprised by their solutions. The only subject who did not rigidly follow her perception was the one who flexibly applied compensation and negation to the algebraic tasks.

As might be expected, since the subjects were required to take basic algebra because they failed an exemption test, the subjects lacked confidence in their algebraic ability. Evidence of this lack of confidence abounded despite the fact that the subjects were successful in a basic algebra course. Three subjects explained their lack of confidence in terms of their tendency to follow rules blindly, unsure of why they were doing what they were doing and whether it would work.

The subjects expressed some definite attitudes toward several of the topics and operations of algebra, but these attitudes varied. The following topics were "hated" by at least two subjects: variables as exponents, radicals, complex numbers, and division. The reactions to division were particularly interesting since the subjects who expressed their feelings about division did so on tasks they solved by multiplication. Yet they did not indicate any strong feelings about tasks involving multiplication. The subjects' initial perception of the task elicited a negative attitude towards the task.

## Classification Test, Mathematical Memory Test, and Preference for Algebraic Problems Test

Each subject used the task's style of presentation in classifying the 21 tasks of the Classification Test. One subject separated the polynomial and algebraic fractions by style of presentation, but separated radical expressions tasks of all types and complex number tasks of all types into separate categories. Another subject separated the tasks first on the basis of style of presentation and then on the basis of operations. The next subject separated the tasks first on the basis of style of presentation, and then on the basis of types of algebraic expressions. And the final subject separated the tasks solely on the basis of style of presentation.

The Mathematical Memory Test met with limited success as a measure of the semantic versus syntactic information that the subjects recalled of problems they had read but had not recently solved. Three subjects could recall something for only the first circled cards. For that card, two of these subjects correctly identified the algebraic topic, operation, and style of presentation and could recall only a few of the syntactic features of the task. The third subject recalled the algebraic topic and the style of presentation correctly, but she recalled the operation (subtraction) as an equal sign. The fourth subject also remembered this task as equating two fractions rather than indicating subtraction. Besides this error and one other, she recalled the task exactly as it was written. She also was able to provide syntactic details, the style of presentation, and the algebraic topic for the two other circled cards, but was unable to name the operation correctly for one of these cards.

The subjects' preference for solving the tasks of the Preference for Algebraic Problems Test differed from their scheme for classifying the tasks. Two subjects used style of presentation as their first criterion in ordering the tasks and algebraic topic as their second criterion. The third subject used operations as the first criterion and style of presentation as the second criterion for ordering the tasks. The fourth subject ordered the tasks first by algebraic topic and then by operations.

# CHAPTER VI

## SUMMARY AND DISCUSSION

The investigation described in this report was a long-term ascertaining study of the understanding of algebra by students who were successful in a basic algebra course. Four processes of understanding of algebra were observed: reversibility of reasoning when solving algebraic problems, flexibility of thought when solving algebraic problems, the ability to generalize algebraic processes and operations, and the ability to transfer algebraic processes and operations within the content of algebra. Evidence of the use of these processes was obtained from case studies of four students of basic algebra who were successful in a basic algebra course.

### Significance of the Study

> It is in beginning with the child, in studying his reactions, in analyzing his reasoning and his methods of resolution, that we will be able on the one hand to explain the reasons of his mathematical incomprehension, and on the other to furnish a scientific response to so controversial a question as knowing when and how we teach arithmetic and algebra. (Johannot, 1974, p.12)

Algebra has traditionally been the mathematics course in which success or failure determines whether a student can go on to higher level mathematics courses or is forever limited to a knowledge of basic arithmetic. Success in learning algebra has to a large extent been based on measures of the student's instrumental understanding; measures of the students ability to recall an appropriate rule and apply it correctly. Yet little is known about what a student who is successful in learning algebra "really" understands. Through the analysis of four case studies, the present investigation provides evidence of the cognitive structure of the understanding of students who are successful in a basic algebra course. The study is important for the following three reasons:

1. Understanding of algebra. The study develops and explores students' understanding of algebra in terms of four processes, thus leading to operational descriptors of understanding that can be used by others in the field. The term "understanding" is used frequently by teachers, students, curriculum developers, textbook authors, and the general population, but the meaning applied to the term may differ widely. By focusing on students who are successful on classroom measures of algebraic skills, the present study helps to clarify what those measures do and do not imply about the following four processes of a student's understanding: her ability to reverse her train of thought in algorithmic operations, her flexibility of approach to

solution, her ability to generalize skill and operations, and her ability to transfer algebraic knowledge and skills to related contexts.

2. Teaching of algebra. By increasing knowledge about the meaning students attach to algebraic operations, the study provides a foundation for improvements in the teaching of algebra. Algebra instructors may be assisted in their search for more effective ways to present and explain concepts such as variable, polynomial, algebraic fraction, radical expression, and complex number and skills such as addition, subtraction, multiplication, and division of polynomials, algebraic fractions, radical expressions, and complex numbers. For example, techniques such as including missing-term tasks in the teaching of an operation or having students generalize solutions to more complicated variants may prove to enhance their reversibility and ability to generalize.

3. Methodology. The study presents a method of research not common in the Western world. Many clinical investigations are too structured in their interviews to provide much insight into students' internalized thought processes. The long-term, flexible interviews of the ascertaining experiment make it well suited for the exploration of students' relational understanding of mathematics. The instruments created for the present study may serve as models for future investigators of relational understanding.

## Summary of Procedure

The study, conducted during the 1978 spring term at the University of Georgia, used four extended case studies. These subjects, all female, were rated as "above average" by their basic algebra instructor. The instructors' ratings were supported by the subjects' success on their classroom tests.

The study was divided into three parts: pre-tests and selection of the subjects; interviews; and posttests and collection of descriptive data. In the first part, the subjects completed an information sheet, the CSMS Algebra Test, and the Generalization Test.

In the main portion of the study, each subject participated in an average of ten 45-minute interview sessions. Clinical interviews employing the "think-aloud" technique of the ascertaining experiment were used to investigate the subject's internal thought processes as she solved nonstandard algebra tasks developed for the study. The interviews were flexible in design. If a subject appeared to be experiencing a great deal of difficulty with a task, the interviewer provided hints of increasing specificity. If an interviewer noticed something of interest in a subject's response, the interviewer created new questions on the spot to follow the direction of the subject's thought. The content tested in the interview sessions trailed its study in the classroom by from two to four weeks. The four topics included in the investigation were operations with polynomials, algebraic fractions, radical expressions, and complex numbers.

In the last part of the study, the subjects completed the Classification, Mathematical Memory, and Preference for Algebraic Problems Tests. Additional descriptive data were collected from regular classroom tests on the same content as the interviews and background information from the subject's permanent files.

The analysis of the protocol data was qualitative. Verbal protocols and written responses of the subjects were scrutinized in terms of the questions of interest. The responses for each subject were analyzed for regularities in observable sequences of behavior across tasks. These individual analyses were reported as case studies, which were supported by excerpts from the protocols and corresponding sections of the written work. The verbal and written responses of the four students were also analyzed for commonalities and differences in observed patterns of behavior across subjects.

## The Study in Perspective

The purpose of this study was to uncover information about and gain a greater insight into the extent to which students who are successful in a basic algebra course possess the ability to perform four processes of understanding. This was not a study of the teaching of algebra, but rather of the learner. The variety of observed behaviors for the subjects can not be attributed to one course. Each subject had had a different series of courses preceding basic algebra. No attempt was made to determine if the instructors of basic algebra taught for relational, as well as instrumental, understanding. It is possible to infer from the course tests provided in Appendix E that much of the evaluation for the course was based on instrumental understanding, but to infer, therefore, that this instruction did not attempt to provide "why to" understanding would be unfair. For example, in her solution to the complex number task Lynn pointed out that even though her instructor told "why" they could use a given formula for multiplying complex numbers, she did not listen because she already knew "how" to apply it.

The present investigation did not support Thorndike's (1926) view that the higher level thought processes will build up over a period of time, although it could not totally discount it. The subjects included in this investigation had each studied algebra for at least two years prior to the study. In their basic algebra classes they demonstrated that they knew how to apply the operations and processes that they had also studied in high school, but they varied greatly in their relational understanding. To the meaningful learning theorist the variation provides evidence that the creation of bonds was not sufficient to develop higher level thought processes; but, on the other hand, the connectionist might argue that either the variation was evidence that these processes were in a state of development or that the bonds had never been properly formed.

The latter view was supported by Shevarev (1975) in his study of the errors made by algebra students who knew a rule and yet at times acted contrary to it. Shevarev analyzed the students' errors in terms of incorrect bonds formed in situations where the students were correctly solving algebraic expressions.

Brownell's (1935) Meaningful Learning Theory was applied to Skemp's (1976) characterization of instrumental and relational understanding as a theoretical background for the present study. The present investigation provided evidence of the invalidity of the use of tests designed to measure instrumental understanding as predictors of relational understanding. Three subjects participating in the present investigation were enrolled in the same section of basic algebra and hence, each of their chapter tests contained the same questions. Several of their responses to the items on the test appeared very similar, suggesting that they had apparently used the same steps to solve the items. The case studies revealed that these same subjects differed in several of their processes of relational understanding.

Based on the analysis of the case studies, only one of the four process of understanding under investigation could be implied by a student's success in a basic algebra course. This was the ability to generalize a known instance to a class of algebraic skills. The students' solutions to the Generalization Test showed that they were able to generalize a given rule to increasingly more complicated variants of the rule even when they had never previously seen the rule. On the algebra interview tasks the students were also able to generalize their solutions to other unknown variants for the task once the task was solved.

## Reversibility and Flexibility

Collis (1975a, 1975b) described the order in which students develop an ability to reverse their mental operations in terms of finding the value of "y" in the equation "y + 4 = 7." Collis suggested that the ordered stages of students' reversibility were compensation, negation by restating the problem as one involving the inverse operation, and negation without upsetting the existing state of equality. The results of the present investigation suggest that Collis's ordering may be more a product of instruction than of the natural development of the student. The subjects in the present study varied widely in their reversibility of algebraic operations. As an apparent contradiction to Collis's ordered stages, one subject solved the tasks by negation but was unable to apply compensation to most of the tasks. The only tasks in which all subjects applied compensation were multiplication tasks with neither factor given. On these tasks the students applied the learned compensation technique of factoring.

In terms of the behaviors of reversibility of multiplication and factoring of polynomials reported by Krutetskii (1976), the subjects in the present study would be described as average. Perhaps instruction in both compensation and negation is needed for students to become flexible in reversing their train of thought on algebraic operations.

## Generalization and Transfer

The students in this study tended to view the learning of each topic of algebra as separate, disjoint units of knowledge to be learned. Each new topic was seen as a series of new rules to learn. The completion of a topic heralded the end of the necessity for holding the rules in memory. Within two to four weeks after a student had demonstrated successful mastery of the processes and operations involved in a particular topic, she would forget having studied how to apply the processes or even that she had studied the topic. Rules such as whether to add or multiply exponents in a particular situation, although finely tuned for a test, were applied uncertainly a few weeks later.

Rather than view the operations for radicals as generalizations of the operations for polynomials, the students perceived the two topics as separate. In applying the algorithms for radicals, the students had difficulty transferring the operations and processes they had learned for polynomials.

The students apparently placed tasks in a particular "frame," as described by Davis and McKnight (1979), and operated without linkage to other frames. For example, once the students perceived a fraction as an indicated division, they would not apply the rules for fractions that they had previously mastered. If students do place tasks in inflexible frames, then instruction must be designed in such a way as to make the students' frames interlock.

Matz (1979) and Kieran (1979) have identified the duality involved in the solving of equations as a cause of many students' errors. Perhaps the use of algebraic variants of "missing addends" tasks, similar to the algebraic interview tasks used in the present study, would provide such a link for the study of algebraic operations and the study of algebraic equations.

Johannot (1974), Matz (1979) and Carry, Lewis, and Bernard (1980) have suggested that intuition of a generalization made prior to an algebraic calculation gives it greater influence than that of the proper algebraic calculation. Perhaps this accounts for the fact that in several instances the students included in the present study generalized false, as well as correct, variations of rules they had previously generalized correctly. If so, then instruction may need to be altered (as suggested by Carry et al, 1980) to present the students with heuristics for testing hypothesized generalizations. Students will need to be further encouraged to use the heuristics by a continued emphasis on solving a variety of tasks.

## Related Abilities and Concepts

In the present study, students' perceptions of algebraic tasks and their attitudes towards the tasks were shown to have a great effect on, as well as be affected by, their ability to make the correct generalizations and transfers. As tasks became more frustrating the students were more likely to make errors. These errors were frequently false generalizations or false transfers of rules the students had previously used correctly. The errors noted could be explained in terms of Shevarev's (1975) analysis of errors that students make because of an incorrect perception of the task. Shevarev found these errors were created by perceptions that were too general, too specific, or too focused.

The basic algebra course, in which the subjects in the present study were enrolled was organized in a mathematically logical sequence. In the textbook, the chapter on polynomials was followed by one on algebraic fractions, which was followed by one on radical expressions and complex numbers. Mathematically, a complex number is a "special" polynomial. Many of the processes and operations used with polynomials can be generalized for radical expressions and complex numbers. For students who were successful in a basic algebra course this process was one of transfer. Each topic of algebra had its own set of operations and processes, its own generalizations. In their classification of algebraic tasks, three of the the four students participating in the study created separate categories for radicals and complex numbers. Although they separated tasks first on the basis of style of presentation of the task, they generally did not include radicals and complex number tasks in these piles. Further, in solving the algebra interview tasks two of the subjects had difficulty adding complex numbers and required hints to associate the operation with addition of polynomials.

Matz (1979) suggested that students need training in goal-oriented approaches to the learning of algebra. Students often have trouble determining which manipulations will be productive and which will be unproductive. Matz conjectured, "This may be a consequence of feeling that algebra involves undirected manipulation of meaningless forms" (p.16). Carry, Lewis, and Bernard (1980) cite a striking example of students' lack of a knowledge of the goal of equation solving. In attempting to solve equations some students would continue to operate on numbers as usual even after the unknowns had vanished from their problem. Perhaps training in the use of planning heuristics would help the students to organize a goal-oriented search in their solution. A similar finding was noted in the present study. Once started on a task the student would loose track of the original problem. This occurred frequently in multiplication tasks involving radical expressions or complex numbers. The students who solved these tasks by negation would rewrite the tasks as fractions and rationalize the denominators. Even when their original task had been resolved as a step in simplifying their fractions, these subjects would continue their simplification process.

The students in the present study were among the successful students in their classes, yet they lacked confidence in their abilities. Three subjects hesitated to check their tasks and then asked for reassurance that their checks were correct. For them a check appeared to be as meaningless a procedure as the one they originally used to solve the task. Two explanations are that since the operations were not totally reversible for these students, they were not comfortable with the solutions obtained by another approach, or that they viewed the only possible check for the task as repetition of the procedure they were trying to check. Topics such as radical expressions and complex numbers were viewed very negatively by some subjects perhaps due to the uncertainty with which they approached the topic. Any words connected with division elicited negative comments from one student because of the difficulty she had in learning division in fourth grade.

The present investigation provides some interesting questions for Skemp's (1976) theoretic formulation of instrumental and relational understanding. Students who were successful in a basic algebra course provided contradictory traces of developing instrumental and relational understanding. At times their comments would be heavily tied to instrumental behaviors, yet in the midst of solving the same task they would provide evidence of processes of relational understanding. Also evident was the fleeting nature of the processes of instrumental understanding for the subjects on certain topics. For example, based on the interviews, one might assume that the students had not encountered the topic of rationalizing the demoninator of a fraction involving radicals, yet copies of their tests revealed that they had demonstrated an instrumental understanding of the topic only a few weeks earlier. Must instrumental understanding precede relational understanding? How stable are instrumental and relational understanding over time?

## Limitations of the Study

There are several obvious limitations to the present study. The small sample size does not permit generalization to any population. All subjects were college basic algebra students and lacked confidence in their mathematical ability, therefore the regularities observed give no evidence that high school students who were successful in a basic algebra course would exhibit the same evidence of relational understanding. The subjects were all volunteers and not randomly selected, which limits any generalization even to other basic algebra students.

Although no instruction was intended, the long term nature of the ascertaining experiment and the flexible use of hints provided instruction for the subjects. The interviews may be developing relational understanding by attempting to observe it. Furthermore, the interviews were not consistent in the order questions were asked. Each subject was presented the tasks in a different order. The subjects' style of solution may have been influenced by the order of presentation.

There were also limitations resulting from the "thinking aloud" method of determining the processes used by the subjects in solving problems. The method may be unreliable since subjects tend to fall into silence during their deepest thoughts. Further, as the problems become routine the subjects shorten their verbalization of their thinking process. Moreover, the method may not be valid since the subjects may subconsciously be "reading" the interviewer and attempting to respond in the manner which they feel that the interviewer expects.

Another limitation of the study was the minimal training which the interviewers underwent before beginning the study. Each interviewer had his or her own style for directing the interview, questioning, and presenting hints. Different interviewers may have yielded a different pattern of behavior.

Finally, the presentation and analysis of the protocols had several limitations. Only the experimenter reviewed each protocol, answered the questions of interest in light of that review, and selected those portions of the protocols to present in support of the analysis. The validity of this analysis might have been greatly improved by having a research team analyze the protocols.

## Suggestions for Future Research

Studies of the learning and teaching of mathematics beyond the elementary school have tended to rely on measures of instrumental understanding as indicators of the students' future success or of the comparative success of a particular instructional methodology. The criteria of rate and accuracy have been applied in these studies, but rarely have the criteria of changes in the level of process, transfer, or retention been applied.

The higher level processes of reversibility, flexibility, the ability to generalize from a given instance to unknown variants, and the ability to transfer mathematics are processes of understanding that can not, at the present state of knowledge, be accurately interpreted with product measures. The investigation of dynamic processes of understanding calls for dynamic research methodologies. Two types of methodologies appropriate for future investigations of the processes of relational understanding are the ascertaining experiment and the teaching experiment.

## Ascertaining Experiments

The ascertaining experiment is particularly well suited for the investigation of relational understanding. By using long-term case studies with flexible questioning, educational researchers will be able to extend and clarify our knowledge of students' relational understanding. Many series of ascertaining studies need to be conducted. The investigation of relational understanding should be extended to studies with different:

1. *Processes of relational understanding*. The present study has focused on four processes of relational understanding. More detailed case studies of the processes investigated in this study should be conducted. Students were found to have their own individual styles with regards to the way in which they attempted to reverse mental operations. Are these styles consistent across the content of algebra? If an algebra student solves most tasks of the form $a * \square = c$ by compensation, where a and c are given polynomials, * is a given operation, and $\square$ is an unknown polynomial, how will they solve linear equations. How flexible is their approach to solving linear equations? Do they transfer their processes and operations for algebraic expressions to algebraic expressions contained in equations? What effect does the duality of the equal sign have on the use of false generalizations and transfers in solving equations and inequalities? Other processes of relational

understanding such as the ability to curtail the reasoning process might also be examined.

2. Ages and levels of mathematical experience. The present study was limited to students at the college level who had had at least two years of high school algebra. This study should be replicated with long-term case studies of the relational understanding of beginning algebra students. The students who participated in the present study were not consistently successful in their high school mathematics courses. What evidence of a relational understanding of mathematics would be provided by the top mathematics students at different stages of mathematical maturity?

3. Less successful students. The present study was aimed at students who had previously demonstrated success in a basic algebra course. The only subject to demonstrate complete reversibility on the algebra interview tasks received a B in her basic algebra course. Perhaps there are students who would score low on measures of instrumental understanding and yet possess well developed processes of relational understanding. How do students rated as above average, average, and below average compare on measures of relational understanding?

4. Multi-year case studies. The study of relational understanding is the study of dynamic processes. How do these processes develop throughout students experience with algebra? Long-term case studies conducted over several years would provide valuable insight into the dynamic

quality of relational understanding. How stable are the processes over time? How do processes change with changing mathematics content?

Another area would be the investigation of the concurrent development of instrumental and relational understanding. Must instrumental understanding be obtained before the processes of relational understanding can develop? Research questions on the concurrent development of the two kinds of processes may best be resolved through the application of teaching experiments.

## Teaching Experiments

Once a body of information is available on students' understanding of algebra, then teaching experiments should be designed to use the normative data to assess the success of experimental instruction. If the development of reversibility and flexibility of algebraic processes and the ability to generalize and transfer algebraic processes and operations is a goal of mathematics education then the success of instructional programs must be assessed in terms of the development of these processes of relational understanding.

Adi (1978), Kieran (1979), and Davis and McKnight (1979) have each proposed instructional methodologies aimed at developing higher level processes of understanding. As educators continue to develop new instructional techniques for increasing students' levels of relational understanding, teaching experiments should be employed to evaluate the success of each instructional technique. Rather than compare one method of instruction with another, as is common in Western research, analysis of a teaching experiment is based on the development of the processes under investigation during instruction. Applied to the development of the processes of relational understanding a teaching experiment might involve the development of a new instructional technique; for example, approaching the instruction of algebra through the development of problem solving. A sample of students might be selected from a class to participate in regular interviews in order to "catch" the development of processes of learning. Instruction would be constantly altered based on the long-term interviews in an effort to develop the instructional methodology to its optimal level. As comparative data the research might compare the processes of relational understanding with the normative data gathered during the ascertaining experiments.

Other teaching experiments might involve studying the instructional techniques of teachers whose students are known to develop high levels of relational understanding. In this case the experimenter might compliment clinical interviews with students with observations of the classroom setting in an effort to gain more insight into the teacher's instructional processes.

## Conclusion

This study began with the question, "What does it mean to say that a student understands algebra?" Two types of understanding were recognized: instrumental ("how to") understanding and relational (knowing why, as well as, how) understanding. Evidence of the level of four processes of understanding were sought in four case studies of college students who were successful in a basic algebra course. The four processes of understanding investigated were reversibility, flexibility, the ability to generalize, and the ability to transfer within the content of algebra.

Analysis of the case studies revealed the following:

1. These students who were successful in a basic algebra course varied widely on measures of four processes of understanding. Tests based on measures of instrumental understanding did not accurately reflect a student's relational understanding.

2. These students possessed definite styles of reversibility. Three of the four students were inflexible in their application of reversible procedures. Since textbook instruction is often focused on the application of negation to the solution of equations, the students' style of reversibility may be contributing to their later difficulty in making algebra meaningful.

3. These students possess many of the same error patterns as do less successful students. These error patterns appear as false generalizations and false transfers

in situations where the task is perceived of as being complicated or placed in a "frame" in which appropriate rules are not applied.

Mathematics educators have long stated the importance of developing an "understanding" of mathematics. The present study provides an exploration of that elusive term.

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# Appendix A

# THE PILOT STUDIES

## First Pilot Study

In the fall of 1977 a pilot study was conducted to develop and test instruments for measuring basic algebra students' ability to generalize, their classification schemes for algebraic tasks, and their memory for algebraic material. The generalization task was adapted from two tests used by Krutetskii (1976) in a teaching experiment with high school students who had not previously learned the rules used in the test. The classification test and mathematical memory test were based on instruments developed by Silver (1978) and Kantowski (1977) in investigating the problem-solving processes used by high school students in solving word problems.

The instruments were tested with 13 students from two sections of a course in basic algebra at the University of Georgia. The students were asked to complete the instruments immediately after turning in their final examinations for the course. The students who volunteered to complete the instruments tended to be among the higher ranking students in each section. The study was designed to investigate the feasibility and value of using the tests with basic algebra students in college. The feasibility was determined by the variety of response patterns. The value was determined by the degree to which the tests could be used to describe students' ability to generalize algebraic rules to presented instances (much like a page of exercises from a text), to provide an indication of the internal classification schemes used by students to organize the content and processes of algebra, and to provide information on the syntactic and semantic aspects of problems which students recall from problems seen but not solved.

Another feature explored in the study was the method of presenting the instruments. The presentation modes used in the first pilot study varied in the experimenter's and student's roles during and after the completion of each instrument. In eight cases the students completed the instruments as they had taken their final exams, without observation by the experimenter. In two cases the students thought aloud as they completed the instruments, and the experimenter observed their actions. After completion the experimenter directed the student to retrospect on his solution process. In the last three cases the students completed the tasks silently, and the experimenter observed their actions. Afterwards, the experimenter directed the student to retrospect on his solutions.

The last was found to be the best method of presentation. In instances where the experimenter was not present some of the students did the generalization task without using the rules provided. For example, instead of using the rule $(a + b)^2 = a^2 + 2ab + b^2$, the students used the FOIL method or another algorithm for calculating the requested products. Although it proved beneficial to have the experimenter present during the administration of the task, due to the algorithmic nature of the task, the "thinking aloud" technique did not provide additional information. Students tended to say exactly what they wrote. Guided retrospection was required to tap the students' thought processes.

The following is a summary of the results of the first pilot study:

1. Krutetskii's generalization tests and Kantowski's and Silver's classification and mathematical memory tests are adaptable for use with basic algebra exercises with college students.

2. The students' responses on the adapted instruments were varied, but there was consistency in their variations. The items on the generalization task were fairly well ordered in level of difficulty as determined by the pattern of student errors. The responses were not random, yet differed from student to student. The response patterns could serve as descriptors of a student's ability to generalize, mathematical memory, and classification scheme for algebraic exercises.

3. The instruments are best administered with the student working on the tasks while being observed by the examiner. Immediately after completing the tasks the interviewer should direct the student to retrospect on his solution process.

## Second Pilot Study

The second pilot study was conducted during the winter of 1978. Its purpose was to examine college students' understanding of basic algebra as it developed during a course based on problem-heuristic instruction. The study was conducted as a Soviet-style teaching experiment with one section (34 students) of basic algebra taught by the investigator. As exemplified in the research of Shapiro (cited in Krutetskii, 1976), the teaching experiment involves the researcher in teaching a class and concurrently conducting individual interviews to determine the course of future lessons and the changes to be made during the course of instruction. The procedures of the classroom interaction and individual interviews for the second pilot study are described below.

### Classroom Activities

Classroom instruction emphasized the arbitrary nature of algorithmic processes, the relationship of one topic to another, and the meaning of the algorithms of the text. Rather than introduce a topic with a general rule, a topic was introduced with a problem. In the beginning of the course the problems were taken from the exercises at the end of the textbook section under discussion. A sample outline for a class lecture is as follows.

$W^n \cdot W^{n-1} \cdot W^3 = ?$

> What should the answer look like? What does $W^n$ mean? What does $W^{n-1}$ mean? What does $W^3$ mean? Have you ever solved anything like this before? What makes this problem difficult? Let me choose a problem like this one that would be easier to solve. (One by one pick examples that incorporate the attributes that make this problem difficult) Now let's solve the given problem. What have we done? Product of monomials. Look at another product of monomials $5X^2Y$ and $6X^3Y^2Z$. What makes this problem difficult? Now solve this problem. We have made use of the general rule that $X^n \cdot X^m = X^{n+m}$.

During the quarter (10 weeks) thirty-five assignments were given (a sample assignment is provided at the end of this appendix). Each assignment consisted of a content problem on the topic of the day, content problems on the topics from the last three to five lessons, an optional verbal transfer problem, and exercises from topics of the preceding week. The intent was to provide time for the student to move from treating the tasks as problems to treating the tasks as exercises. Each assignment sheet included from fifteen to twenty exercises from either related or contrasting sections. For example, exercises on adding and subtracting complex numbers were included with exercises on adding and subtracting real numbers (such as $\sqrt{18} + \sqrt{8}$) and exercises on adding and subtracting polynomials. Exercises on adding and subtracting algebraic fractions were included with exercises on multiplying and dividing algebraic fractions.

Evaluations of the students' development were also made in light of the problem-exercise continuum; i.e., assuming that new algebraic skills are first developed via problem solving and only later by applying algorithms. Twenty-five of the homework assignments were graded on the basis of processes applied by students. Two tests were given during the quarter with the material of the week given as content problems and the material of the preceding weeks given as exercises. The final examination was cummulative, but also treated the most recently encountered topics as content problems. Throughout the evaluation students' processes were emphasized.

## Out-of-Class Activities

Six students volunteered for and consistently came to two individual interview sessions each week for seven weeks. These sessions were audio-taped and helped to guide the instruction. The interview component of the study was divided into three phases. The initial two sessions were designed to obtain information about the entering level of students' abilities and experiences. The final two sessions were designed to obtain final measures of the students' abilities. The rest of the interviews were dynamic, attempting to catch changes in the development of abilities. These sessions were divided into: the experimenter presenting one-to-one trial runs of the lesson for the day, students thinking aloud as they attempted problems and exercises from the next assignment sheet, and students reflecting on the solution processes they exhibited in completing assignments and tests. A copy of the interview outline for the third interview session is included at the end of this appendix.

The information from these interviews demonstrated that verbal content tasks were problem situations for the great majority of students. Although several students treated factoring $9 - 4X^4$ as an exercise before the topic was introduced in class, "finding two binomials whose product is $6X^2 - 13XY + 6Y^2$" was treated as a problem by all students interviewed. Rather than treat this as a factoring problem in the algebraic sense, several students treated it as they would have if they had been asked for a prime factorization of a positive integer. That is, they searched for a binomial that would divide $6X^2 - 13XY + 6Y^2$ evenly. It was not immediately apparent to them that they were in fact factoring.

The initial and final sessions of the interviews provided a testing ground for instruments designed to describe the students in terms of selected mathematical abilities. These instruments included revised versions of the generalization test, the classification test, and the mathematical memory test. The generalization test was given in one of the initial sessions. Its purpose was to measure the students' ability to generalize from a given rule to increasingly more complex instances. The rules were ordered so that rules necessary for the application of a given rule, if provided, preceded the given rule. For example,

$(a^m)^n = a^{mn}$ preceded $(A + B)^2 = A^2 + 2AB + B^2$. The items on the generalization task were changed in two ways. First, the last two rules were changed to ordered pair representations for addition and multiplication of algebraic fractions. Second, the rules for both addition and multiplication were given together. The exercises on which the student was to apply the rules for addition and multiplication were mixed between these rules to examine the difficulty of connecting the proper rule to the indicated opeation. The intent of the first change was to tie the rules more closely to those actually used in basic algebra with operations more familiar to the students.

The classification and mathematical memory tests were given in the final interview session. To classify tasks into a meaningful internal structure, the student should have previously attempted to solve the task. The classification and mathematical memory tests were altered to include the types of problems used in the interview sessions and on the homework assignment sheets.

The Lankton Mid-Year Algebra Test was used to measure the entering behavior of the students in the course on algebraic tasks. The test provided evidence of the wide variation in the students' knowledge of basic algebra, but did not provide evidence of the processes they used.

The following is a summary of the results of the second pilot study:

1. For the given age level and content, the "think aloud" method provides a rich source of information on the processes students use in solving mathematical problems.

2. The tasks developed for use in teaching algebra through problem solving were appropriate for use in individual interview sessions for measuring students' understanding of algebra.

3. The use of hints in an interview setting was appropriate for measuring the development of understanding over the period of instruction.

4. The generalization, mathematical memory, and classification tests were appropriate for use in the present study. The Lankton Mid-Year Algebra Exam did not provide an adequate measure of the entering basic algebra students' processes of understanding of algebra.

5. The time necessary to develop a teaching experiment for an entire course of basic algebra was beyond the scope of a dissertation. Several studies are needed to unravel the complex relationship between instruction and the development of understanding. The first plateau is to develop ways of measuring and investigating students' understanding of algebra. This dessertation is a first step toward that plateau.

## Intermediate Algebra Problem Set #14

1) On page 47 in your text* do problems # 6, 18, 22, 24.
On page 51 in your text do problems # 1, 9, 18, 26, 32, 42, 49, 62, 71, 78, 81, 89, 93.
On page 56 in your text do problems #3, 7, 13, 17, 23.

2) Write each of the following fractions as a single fraction and simplify.

$$\text{(a)}\ \frac{\frac{2}{7}}{\frac{5}{7}} \qquad \text{(b)}\ \frac{\frac{2x^2}{7y}}{\frac{5x}{7y^2}} \qquad \text{(c)}\ \frac{\frac{2a^2}{7b}}{\frac{5a}{14b^2}}$$

3) a) Find the algebraic fraction in lowest terms which multiplied by $\frac{x - 2}{x^2}$ yields $\frac{x - 1}{x^2 + x}$.

b) Find the algebraic fraction in lowest terms which divided by $\frac{3x^2 - 12x}{x^2 - 1}$ yields $\frac{2x - 2}{3x}$.

4) a) Find two algebraic fractions in lowest terms with unequal denominators, whose sum is $\frac{7x - 11}{x^2 - x - 6}$.

b) Find the algebraic fraction in lowest terms such that $\frac{8}{x + 2}$ subtracted from it yields $\frac{-8x + 40}{2x^2 + x - 6}$.

5) a) What trinomial multiplied by $3a + x$ yields the product $27a^3 + x^3$?

b) What binomial multiplied by $(3x^n + 2y)$ yields the product $12x^{2n} - x^ny - 6y^2$?

c) Write a binomial $P(x,y)$ and a binomial $Q(x,y)$. What is the product of $P(x,y)$ and $Q(x,y)$?

d) Take the product obtained in 5c and factor it completely.

6) (Bonus Problem) A daughter is 8 years old and her mother is 38 years old. In how many years will the mother be three times as old as her daughter?

*The text for the course was Gaughan, E. *Intermediate Algebra*. Belmont, California: Wadsworth, 1970.

## Interview #3 - Session Plan

1) Introduce next topic with classroom procedure. Emphasis on student responses to questions. Emphasize that the student should talk as they think about things to provide a trace of their patterns of solution.
2) Questions:
    a) What should the answer look like? What are you asked to do?
    b) Are there any related rules to keep in mind?
    c) Have you ever solved anything like this before?
    d) What makes this problem difficult?
    e) Make up similar problems which get exceedingly more difficult.
    f) Solve your given problem.
    g) Can you think of another problem that would be solved the same way and is more difficult than the given one?
    h) What have we been doing? How does this fit in with what we did the last few days?...with things we've done earlier in the quarter?
    i) What topic (kinds of things) do you think we'll do in the next lesson?
3) Answer student questions or sample homework type problems.
4) Have student discuss the problems we did today. What can he/she remember about them?

# Appendix B

# FORMS USED IN THE SELECTION OF THE SUBJECTS

This appendix includes copies of the following forms used in selecting subjects and in acquiring their permission to use the information obtained in the investigation:

1) Call for Assistance
2) A Word of Thanks
3) Consent Form
4) Release Form

## CALL FOR ASSISTANCE

During the next six weeks an indepth study of the processes of thinkng used by students of Basic Algebra (MAT 100) will be conducted. This paper is a call for MAT 100 students to participate in this investigation.

Each participant will be required to meet with an interviewer two hours a week at a mutually acceptable time for the remainder of the spring quarter. Participation (or non-participation) will not directly affect your grade in the course and you may of course terminate your participation at any time during the investigation. All responses made by the participant, written or oral, will remain completely anonymous. The tasks done during the sessions are closely related to the material of Basic Algebra. Each interviewer has many years of experience teaching mathematics. At the end of each session time will be provided for the participants to ask specific questions about the material in their MAT 100 course. It is believed that participation in the study will provide students with a better understanding of the algebra they learn.

Unfortunately, due to time constraints, only ten students can be used in the study. If you would like to be included as one of the participants in the investigation, please sign your name in the appropriate space below. Participants will be contacted within the next few days to set up an initial meeting time.

Thank you for your
cooperation,

Sidney L. Rachlin

Sidney L Rachlin
Principal Investigator

* * * * *

I do not wish to participate the study.

____________________________

I would like to set up an initial interview.

____________________________

Local Phone # ____________

## A WORD OF THANKS

I want to express my thanks to you for your interest in participating in the algebra study.

Fortunately for me, the number of volunteers far exceeded the number of participants who could be included in the study. This meant a system for choosing paricipants had to be developed. My decisions were based on a number of variables. First, the design of my study called for participants with a variety of background attributes. For example, there were limits placed on the number of students who had studied trigonometry in high school. Further the number of participants chosen from any one section of MAT 100 was kept as small as possible. A third component of the decision making process was the difficulty of avoiding interviewer-student conflicts in scheduling. The final factor was perhaps the most pragmatic - the ease of arranging the final interview. The spring quarter seems to be passing by so quickly (Ugh! Midterms already!) that I was anxious to get the interviewing started. This meant that students who were contacted and later had a conflict with the arranged meeting time were not included in the study. I was also unable to include the students for whom my calls were never completed.

I hope this helps to answer the question: "What ever happened to that study I signed up for?"

Once again, I really do appreciate your interest in the study and am sorry that it was not possible to arrange it so that all volunteers could participate.

Sid Rachlin

Sid Rachlin

# CONSENT FORM

## BASIC ALGEBRA STUDY

I understand that the purpose of this study is to learn more about the processes of thinking used by students of Basic Algebra.

I confirm that my participation as a subject is entirely voluntary. No coercion of any kind has been used to obtain my cooperation.

I realize that my participation (or non-participation) in the investigation will not affect my grade in the course.

I understand that I may withdraw my consent and terminate my participation at any time during the investigation.

I have been informed of the procedure that will be used in the study and understand what will be required of me as a subject.

I understand that all my responses, written or oral, will remain completely anonymous.

I wish to give my cooperation as a subject.

Signed: ______________________

Date: ______________________

# RELEASE FORM

## BASIC ALGEBRA STUDY

I understand that the purpose of this study is to learn more about the processes of thinking used by students of Basic Algebra. To aid the researcher in characterizing the nature of the participants in the study I hereby consent to having my permanent files released to Sid Rachlin. It is my understanding that the information which he obtains from the files will be used solely for the purposes of the study and will remain completely anonymous.

I further consent to having Mr. Rachlin discuss my coursework in MAT 100 with my instructor. It is my understanding that this discussion will take place after my final grade in the course has been posted.

Signed: ______________________________

Name (printed): ______________________

Social Security Number: ______________

# Appendix C

# DESCRIPTIVE INSTRUMENTS

This appendix includes copies of the following tests used in describing the subjects.

1) Generalization Test
2) Classification Test
3) Mathematical Memory Test
4) Preference for Algebraic Problems Test

The CSMS Algebra Test was also used as a descriptive instrument. More information on the CSMS Algebra Test may be obtained from Dr. K. Hart, CSMS, Chelsea College, 88-99 Lillie Road, Fulham, London SWG 7SR, England.

BASIC ALGEBRA STUDY Name ____________________

GENERALIZATION TEST Instructor ________________

Directions: You are asked to solve the problems in this part using the rule provided in each exercise. Do not be alarmed if the material seems unfamiliar, difficult, or meaningless. Make an honest attempt to follow the directions, answer all of the questions, and solve the problems that are presented. Do all work on this paper. DO NOT ERASE. If you made an error just place a slash (/) on your paper after the error and proceed.

RULE 1: $(a^n)^m = a^{nm}$

Apply RULE 1 to write equivalent expressions without parentheses, simplify when possible.

1) $(2^2)^2 =$

2) $(x^2)^4 =$

3) $(b^3)^n =$

4) $(d^x)^5 =$

5) $(5^x)^2 =$

6) $(a^{x+y})^2 =$

RULE 2: $(A + B)^2 = A^2 + 2AB + B^2$

Apply RULE 2 to find the square of the following expressions, simplify when possible.

1) $(x + y)^2 =$

2) $(x + 2)^2 =$

3) $(5x + y)^2 =$

4) $(a^3 + b)^2 =$

5) $(2x^4 + y)^2 =$

6) $(5 + 3x^2)^2 =$

7) $(a^n + b)^2 =$

8) $(x^{3n} + 1)^2 =$

RULE 3: $(a, b) + (c, d) = (ad + bc, bd)$

Apply RULE 3 to find the sum of each of the following, simplify when possible.

1) $(x, y) + (w, z) =$

2) $(2, 5) + (3, 10) =$

3) $(x, 3) + (2, x) =$

4) $(x, x) + (3, 5) =$

5) $(1, a^3) + (3, a^2) =$

6) $(2, x^n) + (6, x^n) =$

7) $(a, a + b) + (b, a + b) =$

8) $(\frac{1}{2}x, x^n) + (x^n, \frac{1}{2}x) =$

RULE 3: $(a, b) + (c, d) = (ad + bc, bd)$

RULE 4: $(a, b) \cdot (c, d) = (ac, bd)$

Apply the appropriate rule to perform the indicated operation and simplify when possible.

1) $(x, y) \cdot (w, z) =$

2) $(2, 5) \cdot (3, 10) =$

3) $(x, 3) \cdot (2, x) =$

4) $(2, 3a^2) \cdot (3, 2a^3) =$

5) $(\frac{1}{2}x, x^n) \cdot (x^n, \frac{1}{2}x) =$

6) $(3a, ac) + (2a, bc) =$

7) $(a, a + b) \cdot (b, a + b) =$

8) $(3a + 2b, a) + (2a, bc) =$

9) $(5x^3, 10x) + (2x^2, 5) =$

10) $(x^3, 3x)^2 =$

# BASIC ALGEBRA STUDY

## CLASSIFICATION TEST

Directions:

Attached to this sheet is an envelope containing twenty-one 3-by-5-inch note cards on which algebraic expressions have been written. In the right hand corner of each card is a blank. On the first card the number "1" has been placed in the blank and circled. Make a pile of all those cards which contain algebraic expressions LIKE the one on the first card. Once you are satisfied that you have all cards LIKE the first card on one pile, place a "1" in the blank in the right hand corner of each card in the pile.

Next, pick any one of the remaining cards and place a "2" in the blank in the right hand corner. Circle the "2". Make a pile of all those cards which contain algebraic expressions LIKE the one on the card on which you circled the "2". When you are satisfied with your selections place a "2" in the blank in the right hand corner of each card in the pile.

Then, pick any one of the remaining cards and place a "3" in the blank in the right hand corner. Circle the "3". Make a pile of all those cards which contain algebraic expressions LIKE the one on the card on which you circled the "3". When you are satisfied with your selections place a "3" in the blank in the right hand corner of each card in the pile.

Continue this process until you have used all the cards in the envelope. Return the cards to the envelope. Then go on to the next page.

DO NOT GO ON TO THE NEXT PAGE UNTIL YOU HAVE RETURNED THE CARDS TO THE ENVELOPE.

Directions: Answer the following after you return the cards to the envelope.

DO NOT LOOK BACK AT THE CARDS.

1. How many different piles did you have? ______

2. Did you have any piles with only one card in it? ______

   How many of these singleton piles did you have? ______

GO ON TO THE NEXT PAGE

## BASIC ALGEBRA STUDY

## MATHEMATICAL MEMORY TEST

1. What features can you remember which you used to classify problems?

2. Tell me as much as you can remember about the cards with the circled numbers in the right hand corner.

# BASIC ALGEBRA STUDY

## PREFERENCE FOR ALGEBRAIC PROBLEMS TEST

Directions:

The twenty-one 3-by-5-inch note cards have been reshuffled. Suppose you were going to be asked to solve each of these problems. Order the cards by choosing the problem which you would prefer to do first. Next choose the card you would prefer to do second. Continue this process until all twenty-one cards have been ordered. You do not need to actually solve the problems, only to place them in your preferred order.

# Tasks Included in the Classification Test, the Matematical Memory Test, and the Preference for Algebraic Problems Test

Tasks are coded by type of algebraic expression [polynomial (P), algebraic fraction (F), radical expression (R), and complex number (C)]; operation [(addition (A), subtraction (S), multiplication (M), division (D), and factoring (F)]; and style of presentation [text (T) and one (1), two (2), or three (3) missing algebraic expressions-to-find].

P - S - T. "Simplify: $A + 2B - (-2B + A)$".

P - S - 1. "What trinomial subtracted from $5X^2Y - XY^2 + 7$ yields the difference $-X^2Y + 8$"?

P - M - T. "Simplify: $(5X - 2)(3X + 1)$".

P - M - 2. "Find two binomials, $M(M,Y)$ and $N(X,Y)$, such that $M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2$".

P - M - 3. "Find two binomials whose product is a binomial. Find the product".

P - D - 2. "Find two monomials whose quotient is $-5AB^2C$".

P - F - T. "Factor: $X^2 + 8X + 6$".

F - A - 1. "Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$".

F - S - T. "Simplify: $\frac{A}{A^2 - 16} - \frac{A + 1}{A^2 - 5A + 4}$".

F - S - 2. "Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$".

F - D - T. "Simplify: $\frac{X^2 - 7X + 12}{X - 2} \div \frac{X^2 - 4X + 4}{3X - 12}$".

F - D - 1. "Find the algebraic fraction in lowest terms if $\frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21}$ divided by it yields $\frac{2Y - 3}{Y - 7}$".

R - A - T. "Simplify: $\sqrt{8} + \sqrt{18}$".

R - A - 1. "Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$".

R - M - 1. "What number multiplied by $2 + 3\sqrt{2}$ equals $-14$"?

R - D - T. "Simplify: $\frac{\sqrt{3}}{1 - \sqrt{3}}$".

R - D - 1. "Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$".

✓ C - A - T. "Simplify: (9 - 3i) + (5 + 2i)".

C - S - 1. "Find the complex number which subtracted from 5 + 3i equals 7 - 3i".

C - M - T. "Simplify: (1 + i)(1 + 2i)".

C - M - 3. "Find two complex numbers of the form A + Bi where $A \neq 0$ and $B \neq 0$, such that their product is a negative integer".

# Appendix D

# ALGEBRA INTERVIEW TASKS

# ALGEBRA INTERVIEW TASKS

## Polynomial Series

1-1 Write a polynomial of degree 6 which is degree 2 in X, degree 5 in Y, and degree 4 in Z.

1-2 Write a binomial in X and Y where one of the terms is a constant. Subtract from your first binomial another binomial in X and Y where one of the terms is a constant. Simplify your expressions by combining similar terms.

1-3 Write a trinomial which contains an $X^2$ in each term and yet contains _no_ like terms.

1-4 Find two binomials whose product is a binomial. Find the product.

1-5 Simplify: $(X + 1)^{1+N} (X + 1)^{1-N}$.

1-6 Simplify the following expressions by finding the indicated products and combining similar terms:

A) $X^N \cdot X^N$
B) $5X^N (6X^N + 3X^2 + 2X^{N-1})$
C) $(5X^N + 2) \ (6X^N + 3)$
D) $(X^N + Y^{N-1})^2$.

1-7 Find two monomials whose quotient is $-5AB^2C$.

1-8 Find two trinomials, $P(X,Y)$ and $Q(X,Y)$, such that

$$P(X,Y) - Q(X,Y) = -2X^2Y.$$

1-9 Find the binomials, $M(X,Y)$ and $N(X,Y)$, such that

$$M(X,Y) \cdot N(X,Y) = 6X^2 - XY - Y^2.$$

1-10 Factor the following binomial into a product of a monomial and a binomial: $A^{N+3} + A^3$.

1-11 Write a binomial $P(X,Y)$ and binomial $Q(X,Y)$. What is the product of $P(X,Y)$ and $Q(X,Y)$? Next, take the product you've just found and factor it.

1-12 What trinomial multiplied by $4A + Y$ yields the product

$$64A^3 + Y^3?$$

1-13 What trinomial multiplied by $A - B - C$ yields the product $A^2 - B^2 - C^2 - 2BC$?

1-14 What trinomial subtracted from $5X^2Y - XY^2 + 7$ yields the difference $-X^2Y + 8$?

1-15 Create a division problem in which a trinomial divided by a binomial yields a binomial and a remainder of 1.

1-16 A) Find the product of $2A + B$ and $3A^2B + AB - C$.

B) Factor the product obtained in part A.

## Algebraic Fractions Series

2-1 Find the algebraic fraction in lowest terms which when added to the sum of $\frac{3X + Y}{Y}$ and $\frac{2X + Y}{X}$ yields $\frac{3X + Y}{Y}$.

2-2 Find the algebraic fraction in lowest terms which when multiplied by the sum of $\frac{3X}{5}$ and $\frac{X - 5}{15}$ yields 1.

2-3 Find the algebraic fraction in lowest terms which divided by $\frac{5X - 1}{X^2 - 25}$ yields the quotient -2.

2-4 Find the algebraic fraction in lowest terms if $\frac{2X + 3}{X^2 - 3X + 2}$ subtracted from it yields $\frac{1}{X^2 - 3X + 2}$.

2-5 Find the algebraic fraction in lowest terms which when added to $\frac{X}{X - 5}$ yields the sum $\frac{2X - 15}{X^2 - 6X + 5}$.

2-6 Find the algebraic fraction in lowest terms which multiplied by $\frac{X - 2}{X^2}$ yields $\frac{X - 1}{X^2 + X}$.

2-7 Find the algebraic fraction in lowest terms if $\frac{X^2 - XY}{XY}$ divided by it yields $\frac{X^2Y}{3}$.

2-8 Find the algebraic fraction in lowest terms which when subtracted from $\frac{8X}{X^2 - Y^2}$ yields the difference $\frac{8Y}{X^2 - Y^2}$.

2-9 Find the algebraic fraction in lowest terms which divided by $\frac{3X^2 - 12X}{X^2 - 1}$ yields $\frac{2X - 2}{3X}$.

2-10 Find the algebraic fraction in lowest terms if $\frac{4Y^2 - 12Y + 9}{Y^2 - 10Y + 21}$ divided by it yields $\frac{2Y - 3}{Y - 7}$.

2-11 Find the algebraic fraction in lowest terms if the sum of $\frac{5X}{X^2 - 1}$ and $\frac{3}{X + 1}$ added to it yields $\frac{17X^2 - 6X - 1}{2X^3 - 2X}$.

2-12 Create a subtraction problem involving two algebraic fractions in lowest terms with the same denominator and whose difference is $\frac{2A}{B}$.

2-13 Create a subtraction problem involving two algebraic fractions in lowest terms with different denominators and whose difference is $\frac{2A}{B}$.

2-14 Find the algebraic fraction in lowest terms if $\frac{X^2 - 4}{X^2 + X - 2}$ multiplied by it yields $\frac{X + 2}{X - 1}$.

2-15 Find the algebraic fraction in lowest terms which when $\frac{8}{X + 2}$ is subtracted from it yields $\frac{-8X + 40}{2X^2 + X - 6}$.

2-16 Find the algebraic fraction in lowest terms which multiplied by $\frac{Y^2 - 7Y + 12}{Y^2 + Y - 2}$ yields $\frac{Y^2 - 3Y + 4}{Y - 1}$.

2-17 Find the algebraic fraction in lowest terms if $\frac{5A + 6}{A^2 + 6A + 9}$ added to it yields $\frac{6A + 9}{A^2 + 6A + 9}$.

## Radical Expressions Tasks

3-1 For a radical to be in simplest form it must be the case that no integral factor can be removed from the radicand. Find the radical equal to $7\sqrt{2}$ which is not in simplest form since an integral factor can be removed from the radicand.

3-2 For a radical to be in simplest form it must be the case that the radical does not appear in the denominator of a fraction. Find a radical equal to $7\sqrt{2}$ which is not in simplest form since the radical appears in the denominator of a fraction.

3-3 What number multiplied by $4\sqrt{2}$ equals $24\sqrt{6}$?

3-4 Find the real number which added to $\sqrt{3}$ equals $\sqrt{6}$.

3-5 What number multiplied by $2 + 3\sqrt{2}$ equals $-14$?

3-6 What number multiplied by the product of $\sqrt[3]{4}$ and $\sqrt[3]{5}$ equals 10?

3-7 What algebraic expression multiplied by $\sqrt[3]{2X}$ equals $\sqrt[12]{128X^{10}Y^6}$.

3-8 Find the real number in simplest form which divided by $\sqrt{2}$ equals $4\sqrt{2}$.

3-9 Find the real number in simplest form if $\sqrt{90}$ subtracted from it equals $3\sqrt{40}$.

3-10 Find the real number which divided by $\sqrt{3}$ equals $3 - \sqrt{3}$.

3-11 Find the real number which multiplied by $\sqrt{2} - 1$ equals 1.

## Complex Numbers Task

4-1 Find the complex number which when added to $-3 + 4i$ equals 10.

4-2 Find the complex number which when multiplied by $3 - i$ equals 10.

4-3 Find the complex number which multiplied by $5 + 3i$ equals $13 + i$.

4-4 Find the complex number which subtracted from $5 + 3i$ equals $7 - 3i$.

4-5 Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ such that their differences is $-2i$.

4-6 Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$, such that their product is a negative integer.

4-7 Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ such that the sum of the two complex numbers is $i$.

4-8 Find two complex numbers of the form $A + Bi$ where $A \neq 0$ and $B \neq 0$ whose quotient is a positive integer.

# Appendix E

# SAMPLE BASIC ALGEBRA TESTS

The sample tests included in this appendix were used in the sections of basic algebra in which the students were enrolled. They provide evidence of the type of tasks on which the subjects demonstrated success in basic algebra. The first three tests were used in the section in which Emily, Lynn, and Claire were enrolled and the last three tests were used in the section in which Patty was enrolled.

Math 100 Name ________________________

Test 1

1. Consider the polynomial $5x^3y^2 - x^2y^3 + 3xy^4$

   (a) State its degree in y. ____________

   (b) State the numerical coefficient of its second term. ____________

2. If $x = 3$ and $y = -2$, evaluate $\frac{4x}{2x + y} - xy$ ____________

3. If $P(x) = x^3 - 3x^2 + 2x + 7$, find (a) $P(0)$, ________

   and (b) $P(-2)$. ________

4. Simplify.

   (a) $5 - [3x - (x - 4) - 1]$ (a) ____________

   (b) $-[(u + 1) - 3(2u - 1) + 16]$ (b) ____________

   (c) $[(x + 2y) - (x - 3y)] - [(x - 2y) - (x + 3y)]$ (c) ____________

For 5 - 11, write each expression as a polynomial in simple form.

5. $(2x^2 + 3x) - (x^2 - 5) + (3x - 5)$ ________________

6. $(x - 3)(2x^2 - 5x + 4)$ ________________

7. $(2x + 3y)(x - 5y)$ ________________

8. $2a^n(3a - 7b^3)$ ________________

9. $(a^n + b^n)(a^n - b^n)$ ______________

10. $(x + y)^2 - (x^2 + y^2)$ ______________

11. Subtract $2x^2 + 3x - 1$ from the sum of $2x^2 - 3x + 5$ and $x^2 + x + 2$. ______________

Simplify. (12 - 14)

12. $(-5xy)\ (2x^2y)\ (-y^3)$ ______________

13. $\frac{12x^4y^7z^5}{-6x^2y^5z^3}$ ______________

14. $\frac{x^ny^{n+2}}{(xy)^2}$ ______________

Factor completely. (15 - 21)

15. $9x^2 - 25$[illegible]$^2$ ______________

16. $y^3 - 9y^2 + 20y$ ______________

17. $15x^2y + 18xy^2 - 3xy$ ______________

18. $x^3 + 8y^3$ ______________

19. $4x^4 - 25y^4$ ______________

20. $y^2 + 2y - 3y - 6$ ______________

21. $20x^2 - 2$[illegible]$b - 21b^2$ ______________

Math 100 Name ______________________

Test 2

1. Give the set of excluded values of the variable:

$$\frac{1}{r^2 + 5r - 14}$$ ______________

2. If $\frac{x}{x-2} = \frac{?}{x^2-3x+2}$ find the missing numerator

______________

3. Divide $z^3 - 8z - 3$ by $z - 3$ ______________

Write in lowest terms (4-5)

4. $\frac{-2xy^2}{24xy} =$ ______________ 5. $\frac{3x^2 - 9x}{x^2 - 5x + 6} =$ ______________

6. Explain why $\frac{3 + y}{y}$ cannot be reduced.

Write each expression as a single fraction in lowest terms.

7. $\frac{1}{2x} - \frac{1}{5y} + \frac{x - y}{2xy}$ ______________

8. $\frac{x}{x^2 - 16} - \frac{x + 1}{x^2 - 3x - 4}$ ______________

9. $\frac{3x}{y} \cdot \frac{2y^3}{9x} \cdot \frac{3y}{4x^2}$ ____________________

10. $\frac{x^3 - 4x}{x^2 + x - 6} \cdot \frac{2x + 6}{x^2 + x - 2}$ ____________________

11. $\frac{10x^3y^2}{3} + \frac{3x^2y^4}{9}$ ____________________

12. $\frac{3x - 12}{2x + y} + \frac{9x - 36}{6x + 3y}$ ____________________

13. $\frac{4}{k^2 - 25} - \frac{2}{k + 5} - \frac{k + 2}{5 - k}$ ____________________

14. $\dfrac{\frac{cy - cz}{y^2 - z^2}}{\frac{y - c}{y + c}}$ ____________________

Math 100 Name ______________________

Test 3

PUT ANSWERS HERE

1. Simplify: $\frac{(-x)^2(-x^2)^3y^3}{(xy)^2}$

a) $x^4y$ b) $-x^4y$ c) $x^5y$ d) $-x^5y$ * 1) ___

2. Which of the following expressions has a value of 2?

a) $(2x)^0y^2$ b) $(2xy^2)^0$ c) $2(xy^2)^0$ d) $2x^0y^2$ 2) ___

3. When $(y + y^{-1})^{-1}$ is written in simplest form the result is

a) $\frac{y}{y^2 + 1}$ b) $y^{-1} + y^{-2}$ c) $\frac{y + 1}{y^2}$ d) $\frac{y^2 + 1}{y}$ 3) ___

4) Simplify: $\left(\frac{a^{\frac{1}{2}}a^{\frac{3}{4}}}{a}\right)^4$

a) $\frac{1}{a}$ b) $a^5$ c) $\frac{1}{a^5}$ d) $a$ 4) ___

5) Simplify: $a^{-\frac{2}{3}}(a^{\frac{2}{3}} - a^{\frac{5}{3}})$

a) $a^{-\frac{4}{3}} - a^{-\frac{7}{3}}$ b) $a^{-\frac{2}{3}}$ c) $a$ d) $1 - a$ 5) ___

* [Error on original test]

6. Write $x^{\frac{3}{2}}$ as a radical.

a) $\sqrt[3]{x}$ b) $\frac{1}{\sqrt[3]{x}}$ c) $\frac{1}{\sqrt{x^3}}$ d) $\sqrt{x^3}$

6) ___

7) $\frac{1}{\sqrt[4]{(a+b)^3}}$ may also be written as

a) $(a+b)^{-\frac{3}{4}}$ b) $(a^3+b^3)^{-\frac{1}{4}}$ c) $(a^3+b^3)^{\frac{1}{4}}$

d) $(a+b)^{\frac{3}{4}}$

7) ___

For 8 and 9, find the indicated root.

8) $\sqrt[3]{-\frac{8}{27}a^3b^{12}}$

a) $-\frac{2}{3}b^4$ b) $-\frac{2}{3}ab^4$ c) $-\frac{2}{3}b^9$ d) $-\frac{2}{3}ab^9$

8) ___

9) $\sqrt[6]{x^7y^8}$

a) $x^6y^6\sqrt{x^5y^2}$ b) $x^5y^2$ c) $\sqrt[6]{x^5y^2}$ d) $xy\sqrt[6]{x^5y^2}$

9) ___

10) Rationalize denominator: $\frac{3}{\sqrt[3]{x^2y}}$

a) $\frac{3}{xy}$ b) $\frac{3\sqrt[3]{x^2y}}{xy}$ c) $\frac{3\sqrt[3]{xy^2}}{xy}$ d) $\frac{\sqrt[3]{3xy^2}}{xy}$

10) ___

11) Simplify: $\sqrt{50xy} - \sqrt{8xy} - \sqrt{16xy}$

a) $3\sqrt{2xy} - 4\sqrt{xy}$ b) $-\sqrt{xy}$ c) $\sqrt{26xy}$ d) $-\sqrt{2xy}$

11) ___

12) $(5 + \sqrt{2})^2 =$

a) $50$ b) $27 + 10\sqrt{2}$ c) $25 + 12\sqrt{2}$ d) $37\sqrt{2}$

12) ___

13) $(2x - \sqrt{y})(2x + \sqrt{y}) =$

a) $4x^2 - y^2$ b) $4x^2 - y$ c) $2x - y$ d) $2x^2 - y^2$

13) ___

14) Simplify: $\frac{3}{\sqrt{2x}} - \frac{1}{\sqrt{x}}$

a) $\frac{3\sqrt{2x} - 2\sqrt{x}}{2x}$ b) $\frac{2\sqrt{3x} - \sqrt{x}}{2x}$ c) $\frac{\sqrt{2x} - 3\sqrt{x}}{2\sqrt{x}}$

d) $\frac{\sqrt{2x} - x\sqrt{3}}{2x}$

14) ___

15) Simplify: $8 + \sqrt{-25}$

a) $-3i$ b) $13i$ c) $8 - 5i$ d) $8 + 5i$

15) ___

16) $(2 + 3i)(5 - 4i) =$

a) $-2 + 7i$ b) $29i$ c) $22 + 7i$ d) $10 - 5i$

16) ___

Math 100 Name ______________________

Quiz #1

Write as the product of primes

(5) 1. 147 =

(10) 2. 1260 =

Insert "<" or ">" to make a true statement.

(5) 3. $\frac{5}{3} \; \square \; \frac{3}{4}$

(10) 4. If $a < b$ and $x < 0$ then $ax \; \square \; bx$

(15) 5. $P(X) = 3X^2 - 4X + 3$, then $P(8) =$

(15) 6. $Q(X,Y) = 2X^2 - 3XY - Y^2$, then $Q(8,-3) =$

Simplify:

(20) 7. $(X + Y)(X + Y) =$

(20) 8. $a^2b(a^2b + 3ab^2 + ab) =$

(20) 9. $(3X + 2Y)(7X - 3Y) =$

(30) 10. $(a - b + c)\ (a + b - c) =$

Math 100 Name ______________________

Quiz #2

(5) 1. $\frac{x^{6n}}{x^4} =$

(10) 2. $8a + (b - 1) + [a - (3b + 16)] =$

(15) 3. $X^{3n} + X^{n+1} - X^2 = X^2x\ (\qquad\qquad\qquad)$

FACTOR:

(10) 4. $8y^2 + 16 =$

(10) 5. $9X^2 - 36 =$

(10) 6. $Y^2 - Y - 36 =$

(15) 7. $3X^2 - 150X + 147 =$

(15) 8. $3b^2 + 11ab - 20a^2 =$

(25) 9. $X^4 - 2X^2 - 8 =$

(25) 10. $12a^2b^2 + 4ab^2 - 8b^2 =$

Math 100 Name ______________________

Quiz #4

All questions count 20 points.

Simplify: Rationalize the denominator when required.

1. $x^{\frac{3}{4}} \cdot x^{\frac{2}{3}} =$

2. $\left(\frac{y^{\frac{1}{3}} \cdot y^{\frac{3}{2}}}{y^2}\right)^6 =$

3. $\frac{x^5}{x^2} =$

4. $\sqrt{18} =$

5. $\frac{\sqrt{a} \cdot \sqrt{a^2b}}{2\sqrt{ab}} =$

6. $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} =$

7. $3\sqrt{5} + 2\sqrt{5} =$

8. $\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{2}} =$

Extra credit (30)

$$\frac{x\sqrt{y} - y\sqrt{x}}{\sqrt{y} - \sqrt{x}} =$$

## Appendix F

## SAMPLE TRANSCRIPTS AND CODING SHEETS

This section includes transcripts and coding sheets for Patty's solutions to Tasks 1-7, 1-8, and 1-9. Due to the time constraints of transcription, the coding sheet evolved as a method for recording notes on the relevance of the solution of each task to the research questions.

**Subject:** Pattie　　　　　　　　　　　**Problem #** 1-7

**Interviewer:** Cherie

**Comments:** Although subject solves the problem by multiplying, she initially views it as a division problem. Subject applies the heuristics of restating the problem, trying a similar problem, devising a plan, and looking back, very naturally. Looking back strategies are of particular note since they are rarely found in students. How stable is its occurence for the subject? For other subjects on these tasks?

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| S201: | Find two monomials whose quotient is $-5AB^2C$. | [Find two monomials whose quotient is $-5AB^2C$] |
| | So, let's see, the first thing I think of is: "OK, find two-one, they're not binomials, they're just one term and when you divide them you get $-5AB^2C$. | [Restating the problem] |
| | Well, I guess... Oh, I hate things like this because I hate division. | |
| | I'm trying to figure out what to do first. Do I ..."whose quotient"... | [Viewed as a division problem, because of the word quotient] |
| | OK when you divide it, it equals that. So something divided by something... | [Attitude already negative before attempting solution] |
| | So this times something..Wait a minute ... "Find two monomials whose quotient is -5AB" ... Alright, I'm doing this.. OK, usually when I do something like | [Rereads problem, restates, attempts generalisation but gets lost so goes back to reading the problem.] |
| | this and I can't figure it out right away, I think of just numbers and I think "OK, find two numbers whose quotient is 5." OK 3, I mean 15 divided by 3 is 5. So if I, I guess | [Uses similar problem with "just numbers"] |
| | what I do is just find...multiply this | [Makes generalisation] |
| | by something and then get the answer and then I'll have two monomials whose quotient is $-5AB^2C$. So I'll just see what happens here. If I multiply this | [Devises a plan] |

**Subject:** Pattie
**Interviewer:** Cherie

Problem # 1-7
Page 2

| Speaker: Counter | Audio | Written [Comments] |
|---|---|---|
| | by ... I'll just do what I just did .. if I multiply it by 3, 3ac or something and just see what happens and I get $-15A^2B^2C^2$. So if I have $-15\ A^2B^2C^2$ and I divide it by 3AC and I divide 3 into 15 and I get -5 and I divide A into $A^2$ and I get A, I leave $B^2$, I divide C into $C^2$ and I get C and the answer is $-5AB^2C$. So, wait a minute ... Yeah, OK. | $(-5ab^2c)$<br>[Builds on similar problem]<br>$(-5ab^2c)\ (3ac)$<br>$-15a^2b^2c^2$<br>$\frac{-15a^2b^2c^2}{3ac}$<br>$\frac{\overset{-5a}{\cancel{-15a^2}}b^2\overset{c}{\cancel{c^2}}}{\cancel{3ac}} = -5ab^2c$<br>[Student naturally checks her solution with the problem - the looking back strategies are naturally employed - Why? Perhaps because of the apparent generalizability of the task.] |
| I237: | How do you feel about your answer? | |
| S238: | So I got two monomials $15A^2B^2C^2$, that's a monomial and 3AC is a monomial and their quotient is $-5AB^2C$. | |
| I240: | Do you think you're right? | $-15\ a^2b^2c^2$ 3ac |
| S241: | I think I'm right. I don't know if I went about it the right way. | [Checks solution with the conditions of the problem] |
| I242: | There's no right way. | [Student is confident in the solution, but questions her own methodology] |
| S243: | OK, the best way. Ha, ha. | |
| I244: | Do you think you could find another set? | [What does the "best way" mean?] |
| S245: | It could be anything because anything you multiply that by and the answer you could just take and when you divide them it's going to be that. So there's endless amounts of things you can do. | [Quickly, generalizes the situation to others of the same specific form. In solving the next problem she shows that this process has been further generalized to one in which a difference, rather than a quotient has been provided.] |

[Time - 3:35]

TASK 1-7 STUDENT P SESSION 2
TAPE COUNTER 201 INTERVIEWER C
CHECK LIST: SOLVED ✓ HINT ___ TRANSCRIBE ✓
COMPENSATION __ NEGATION ✓
UNIQUE __ MORE THAN 1 SOLUTION __ GENERAL SOLUTION ✓

NOTES BELOW REFER TO:

1) FLEXIBILITY ___
2) REVERSIBILITY ✓
3) GENERALIZATION ✓
4) TRANSFER ✓
5) FALSE GENERALIZATIONS ___
6) FALSE TRANSFER ___
7) PERCEPTION AND MEMORY ✓
8) ATTITUDE AND SELF-PERCEPTION ✓

NOTES:

8) The word quotient sets off a reaction: "Oh, I hate things like these because I hate division!"

2) Negation -- Picks arbitrary monomial to multiply by.

4) Transfers process from numbers to polynomials: "OK, usually when I do something like this and I can't figure it out right away, I think of just numbers."

7) Perception of "the right way" and lack of confidence in her own way as being "right" pops up in several tasks.

3) General statement of the solution -- "It could be anything because anything you multiply that by, and the answer you could just take and when you divide them its going to be that. So there's endless amounts of things you can do.

**Subject:** Pattie **Problem #** 1-8

**Interviewer:** Cherie

**Comments:** If student doesn't understand notation, it is explained until both interviewer and the student believe it is understood. Student paraphrases problem, relates this problem to a previously solved problem, plans how to solve the problem, begins to think problem through to what a solution would have to be like, but then decides just to try a value and see what happens, still some confusion about combining similar terms, analyses solution to alter guess, looking back naturally used. Solution process generalized.

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| S248: | Find two trinomials | [Find two trinomials, P(X,Y) and Q(X,Y), such that $P(X,Y) - Q(X,Y) = -2X^{2}$[illegible]] |
| I249: | Do you know that notation? We say P of X, Y. | |
| S250: | Yeah P(X,Y) and Q(X,Y), such that $P(X,Y) - Q(X,Y) = -2X^{2}Y$. Umm, what's P and Q? | |
| I253: | Do you know what that notation means? | [Interviewer defines unknown notation] |
| S254: | No. | |
| I254: | OK, these are just fancy names. I'll write this on the back. If I was to write, say an expression like $X^{2} + X$ or say $3X^{3} + 2X + 5$. This would be what would be called an expression in the variable X and so I might write it like, oh P(X). | $X^{2} + X$<br>$3X^{3} + 2X + 5$<br>$P(X) = 3X^{3} + 2X + 5$ |
| S260: | Is that like using an f(x)? Is that the same thing? | [Student has meaning for f(x) but it has not been generalized to other functional notation.] |
| I261: | It's just like f(x). So for P(X,Y) that means that whatever terms I have I will either have X in it, Y in it, or neither, or a combination. So could you give me an example of one. | |
| S264: | P(X,Y) would be like $2X^{2} + 3XY$ + something. As long as you have both, ... | $P(X, Y) = 2X^{2} + 3XY + 5$ |

Subject: Pattie Problem # 1-6
Interviewer: Cherie Page 2

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| I267: | If you have both, one, but you can't have any other variables. If you're going to use variables, just those two. | |
| S268: | Just those two. | |
| I269: | Is that clear? | |
| S269: | Yeah, I think so. So, find two trinomials using X and Y, only X and Y, such that for the first one minus the second one is equal to that, $-2X^2Y$. OK, well if I wanted to do this one like the other one, I would ... To find two such things, I would take $-2X^2Y$ and add something to it and get an answer and I would add ... I have to add a trinomial. So $-2X^2Y$ + ... I can only use X and Y so I'm going to use + $X^2$ + ... Umm. | [Paraphrase problem]<br>[Similar problem. The process used to solve a multiplication/division problem is being applied to an addition/subtraction problem.]<br>[Devising a general plan]<br>$(-2X^2Y) + (X^2 +$ |
| I283: | What are you thinking? | |
| S284: | I'm just thinking, "Is this going to be really complicated or what?" But I'm just going to write something down and see what happens. $X^2 + 2Y + 4$. And see what that equals. Well, umm, I get -2... | [Student seems at a point of wishing to control more than one condition, but fears or wishes to avoid the complication]<br>$(-2X^2Y) + (X^2 + 2Y + 4) =$ |
| I288: | Tell us what you're thinking. | |
| S289: | OK, I'm trying to remember if I have to ... Actually I don't really need ... Yes I do. Umm this is supposed to be minus first of all. So it's not, it's supposed to be plus 'cause I'm doing it backwards, OK, $-2X^2Y + X^2$ is ... -2 - 1 is just ... | [The fight for whether similar terms are necessary]<br>$(-2X^2Y) - (X^2 + 2Y + 4) =$<br>$(-2X^2Y) + (X^2 + 2Y + 4) =$<br>[While student is frustrated by whether similar terms are needed, the student glances back at the problem statement |

**Subject:** Pattie

**Interviewer:** Cherie

Problem # 1-8

Page 3

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| I294: | What are you thinking? | and momentarily thinks the operation is subtraction] |
| S295: | OK, I'm trying to remember if ... You can't ... OK, I can't add terms that are not like terms. So, actually what this turns out to be is just that; is $-2X^2Y + X^2 + 2Y + 4$. So, that's not a trinomial So what I have to do is make this something that will make it a trinomial. So I could try doing that plus and make it another $-2X^2Y+2Y+4$. And that would be .. You'd combine like terms and you'd have a $-4X^2Y + 2Y + 4$ and that is a trinomial So if you take $-4X^2Y+ 2Y + 4 - (-2X^2Y + 2Y + 4)$. And, let's see, shouldn't you take out the parentheses and that means you change the signs of everything inside of it. $-2Y-Y=$...$-4X^2Y$ plus $2X^2Y$ is $-2X^2Y$. And $2Y-2Y$ is 0 and 4-4 is 0 and so you get $-2X^2Y$. And I used two trinomials and that's what I got. So I think that's right. And I used only X and Y. | [Student views techniques as working backwards]<br>[Solution doesn't meet necessary conditions]<br>[Uses the incorrect expression and changes it to one which will do the job]<br>$(-2X^2Y) + (-2X^2Y+2Y+4)$<br>$-4X^2Y + 2Y + 4$<br>$(-4X^2Y + 2Y + 4) - (- 2X^2Y + 2Y + 4) =$<br>[Student immediately goes to looking back strategies]<br>[A rule: Take out the parentheses and change the sign of everything inside]<br>$-4X^2Y + 2Y + 4 + 2X^2Y - 2Y - 4 = -2X^2Y$.<br>[Checking to see if all the conditions of the problem have been met] |
| I323: | Could you find another pair? | |
| S323: | Well, yeah. Because, this is just like the other one. And there's a million you could do, because you just have to make sure that you have a similar term so the answer's a trinomial. So, if you combine ... The first time I did it I ended up with four terms instead of three. To make it three terms you have to have one of them, that ... be able to combine it with $-2X^2Y$. So, I could. I could just keep changing the constant number in front of it. | [A beautiful generalized statement, pointing out what was wrong in a general way with the first guess]<br>[Time - 6:17] |

TASK 1-8 STUDENT P SESSION 2
TAPE COUNTER 248 INTERVIEWER C
CHECK LIST: SOLVED ✓ HINT ____ TRANSCRIBE ✓
COMPENSATION __ NEGATION ✓
UNIQUE __ MORE THAN 1 SOLUTION __ GENERAL SOLUTION ✓

NOTES BELOW REFER TO:

1) FLEXIBILITY ___
2) REVERSIBILITY ✓
3) GENERALIZATION ✓
4) TRANSFER ___
5) FALSE GENERALIZATIONS ___
6) FALSE TRANSFER ___
7) PERCEPTION AND MEMORY ___
8) ATTITUDE AND SELF-PERCEPTION ___

NOTES:

3) Generalization of process of reversibility from division of polynomials to subtraction of polynomials.

2) Negation -- first chooses a random trinomial, then modifies it to get one that meets the conditions of the problem.

3) Generalized solution -- "Could you find another pair?" "Well, yeah. Because, this is just like the other one. And there's a million you could do, because you just have to make sure that you have a similar term so the answer's a trinomial. So, if you combine -- the first time I did it I ended up with four terms instead of three. To make it three terms you have to have one of them, that ... be able to combine it with - $2X^2Y$. So, I could ... I could just keep changing the constant number in front of it.

**Subject:** Pattie **Problem #** 1-9
**Interviewer:** Sid

**Comments:** Viewing this problem like others caused some difficulty. In the difficulty the student improperly divided - cancelling. This set was easily broken and the student latched on to factoring as a route to solution. After several trials the student found the factors, but consistently multiplied first terms and got a linear term. Student always multiplies vertically. Student unsure of whether other factors were possible, but not found.

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| S000: | Find two binomials M | [Find two binomials, M(X,Y) and N(X,Y), such that $M(X,Y)\cdot N(X,Y)=6X^2-XY-Y^2$] |
| I001: | Of X,Y | |
| S001: | And N(X,Y), such that M(X,Y) times N(X,Y) equals $6X^2-XY-Y^2$. Alright, let's | |
| | see, this reminds me of yesterday when I had the one, two like this, when one was subtraction and one was division or | [Problem is seen as like two that the student has previously solved] |
| | something. So I guess what I will do is | [Devises a plan] |
| | divide $6X^2-XY-Y^2$ by a binomial and hope that I get a binomial for the answer. I guess. Umm, let's see, I will divide it by $2X^2$-... oh, it has to be a binomial. | $\frac{6X^2-XY-Y^2}{2X^2}$ |
| | OK, umm, $2X^2$... Now wait, maybe I'll divide it by ... alright... 2... OK, if I divide it by $6X^2-2Y^2$ then these two are divided out. And that $Y^2$ and | [Checks steps with conditions of the problem]<br>$\frac{\cancel{6x^2}-xy-\cancel{y^2}}{\cancel{6x^2}-2y^2}$ |
| | one of those $Y^2$. So I have: -XY over -2Y. No. Umm, two binomials... using X and Y ... umm, OK, wait a minute. | [Large scale error. Cancelling. Never resolved that this is in error.] |
| | Maybe if, two binomials probably is like trying to factor that. And that would be two binomials I think; so... let's see 2X and 3X. Umm, no, yes, now wait... | [Recognizes that a binomial was still not obtained as an answer]<br>$6x^2-xy-y^2$<br>$(2x-y)\ (-3x+y)$ |
| | -3X. Alright I guess I'll just have to try it and see if it works... No, that's | [Flexible Thought - focuses on getting two binomials and hence views it like factoring.] |

**Subject: Pattie** Problem # 1-9
**Interviewer: Sid** Page 2

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| | not right. Try another one. (3X-Y),(2X+Y) $6X-2XY+3XY-Y^2$ and that's -2+3 is positive XY. So I'll try (3X+Y)(2X-Y) and get $6X+2XY-3XY-Y^2$. And I have $6X-XY-Y^2$ which is that. So there are two binomials and when you multiply them they equal that. | $-3X+Y$<br>$\underline{2X-Y}$<br>$-6X+2XY$<br>[First attempt is aimed at getting -XY by having the 3 be negative.] |
| I051: | OK | $3X-Y$<br>$\underline{2X+Y}$<br>$6X-2XY$<br>$+3XY-Y$<br>[Student consistently views the first term as linear. Even when checking the solution to see if it's correct.]<br>$(3X+Y)$<br>$\underline{(2X-Y)}$<br>$6X+2XY$<br>$\underline{-3XY-Y^2}$<br>$6X-XY-Y^2$ |
| S052: | Now you're going to ask me if I can do it another way. Right, Ha, ha. I guess. | [Student guesses question to to be can she do it another way. Following pattern of questioning used.] |
| I053: | That sounds like a good question. | |
| S054: | Darn, ha, ha. Umm. Well. I don't know if I would do it another way. The other ones you could because you could pick any number you wanted to, like in the one that said find two terms whose quotient equals this. You could multiply the quotient by anything and get anything and those were the two terms that you could use. Only this one, you know, I don't know if there's... How many factors there are. I guess if I sat here I ... maybe I could find two more ... maybe ... But | [Student discusses how this problem differs from the ones she viewed this as like.]<br>[Student does not realize what a factorization yields] |

**Subject:** Pattie  **Problem #** 1-9
**Interviewer:** Sid  **Page** 3

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| | I don't know. | |
| I065: | What are you thinking? | |
| S066: | Well I'm trying to think, how else to find two other numbers. I mean, besides ... you just factor it. I don't ... I don't know. | |
| I069: | OK, suppose you factor it again. Could you find two other ones factoring it again? | |
| S070: | Just this same way. Just ... | |
| I071: | Uh huh. | |
| S072: | Well, let's see. I don't know. I ... you could try, but I don't think so. Because the only other way you could do this would be 6 and 1 and you couldn't get a negative 1. Or could you? Well... | [Student is not confident in this area.] |
| I080: | Watcha thinking? | |
| S081: | Well, maybe I should just do some and see if they work. They must be (mumbled) to work. | |
| I085: | There are lots of things that you're thinking about Pattie and I don't want you to describe what you did think, but try and say things as they're popping into your mind. | |
| S086: | Well, I keep, cause I think ... You know, Ok, I'm going to multiply (6X-Y) times just, you know it would be, (1X+Y). But I, I just don't think that it's going to work. Because I don't see how you're going to get -XY. So. | |
| I090: | OK, that's a good point. | |

Subject: Pattie | Problem # 1-9
Interviewer: Sid | Page 4

| Speaker Counter | Audio | Written [Comments] |
|---|---|---|
| S091: | So now, I don't ... Well, what else can I do? How else do you factor something? And I'm thinking that I don't know. | |
| I095: | Ok, nobody's asked you that before. | |
| S096: | That's true. | |
| I097: | So that's why. Let me digress from that question and go to a different question. Originally you tried to solve it a different way. And then you dropped the idea. Why'd you drop it? | |
| S098: | Because it was getting too messed up. And then I remembered there was an easier way to do it. | |
| I099: | I see. | |
| S100: | Ok. And the reason I did it that way first is because that's what I did in the two before it. You know, it's that little trick, called a mind set, or something like that. So.... | [The reason student first tried division was because of the successful solutions for two previous problems.] |
| I102: | Good, Ok, let me answer my own question to you then and go on to a new problem. Those two factors are unique. Where the other ones weren't. And it's sort of like when you get a problem like asking you to factor 12 over the primes. And no matter how you go about doing it, when you're finished you're going to end up with 3x2x2. And this is the same way. These are prime factors because they can't be factored and so there's no way to get other factors for doing that. | [This explanation is more for the interviewer than the student. It generates little interest.] |
| S109: | Good, Ok. | [Time - 9:28] |

TASK 1-9 STUDENT P SESSION 3
TAPE COUNTER 000 INTERVIEWER S
CHECK LIST: SOLVED ✓ HINT ___ TRANSCRIBE ✓
COMPENSATION ✓ NEGATION ✓ 1st
UNIQUE ✓ MORE THAN 1 SOLUTION __ GENERAL SOLUTION __

NOTES BELOW REFER TO:

| | | | |
|---|---|---|---|
| 1) FLEXIBILITY | ✓ | 5) FALSE GENERALIZATIONS | ✓ |
| 2) REVERSIBILITY | ✓ | 6) FALSE TRANSFER | |
| 3) GENERALIZATION | ✓ | 7) PERCEPTION AND MEMORY | ✓ |
| 4) TRANSFER | | 8) ATTITUDE AND SELF-PERCEPTION | |

NOTES:

7) Initially the task is not seen as asking her to factor.

3) Task is seen as like two previous problems and attempts to generalize the solution process to multiplication of polynomials.

2) First tries negation: dividing by an arbitrary binomial.

5) Since she views this as dividing and "not a fraction", she "divides out" $\frac{\cancel{6X^2} - XY - \cancel{Y^2}}{\cancel{6X^2} - 2Y^{\cancel{2}}}$ : "OK, if I divide it by $6X^2 - 2Y^2$ then these two are divided out. And that $Y^2$ and one of those $Y^2$. So I hope: -XY over -2Y. Drops answer because it is not a binomial.

1&2) Switches approach to factoring (compensation).

5) Consistently misses writing the "square" on the first term -- "$6X - XY - Y^2$".

7) And now you're going to ask me if I could find any other answers. I don't know how many factors there are. I guess I could find two more, maybe, but I don't know. I'm trying to think of how else to find two others unless you factor.